

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Notes about the integral representation of the hierarchical surplus

Theorem:

For a function u with bounded second derivative, we can write its hierarchical surpluses as:

$$v_{l,i} := u(x_{l,i}) - \frac{1}{2}(u(x_{l,i-1}) + u(x_{l,i+1})) = \int_{\Omega} \psi_{l,i}(x) u''(x) dx; \quad \psi_{l,i}(x) = -\frac{h_l}{2} \phi_{l,i}(x)$$

Proof:

$$\begin{aligned} v_{l,i} &= \int_{\Omega} -\frac{h_l}{2} \phi_{l,i}(x) u''(x) dx \\ &= -\frac{h_l}{2} \left(\underbrace{\int_{x_{l,i-1}}^{x_{l,i}} \phi_{l,i}(x) u''(x) dx}_{=: A} + \underbrace{\int_{x_{l,i}}^{x_{l,i+1}} \phi_{l,i}(x) u''(x) dx}_{=: B} \right) \end{aligned}$$

(Note: $\phi_{l,i}(x)$ a linear function in $[x_{l,i-1}, x_{l,i}]$ and $[x_{l,i}, x_{l,i+1}]$, resp.)

$$\begin{aligned} A &= [\phi_{l,i}(x) u'(x)]_{x_{l,i-1}}^{x_{l,i}} - \int_{x_{l,i-1}}^{x_{l,i}} \phi'_{l,i}(x) u'(x) dx && \text{(partial integration)} \\ &= u'(x_{l,i}) - h_l^{-1} \int_{x_{l,i-1}}^{x_{l,i}} u'(x) dx && \text{(as } \phi'_{l,i}(x) = h_l^{-1} \text{)} \\ &= u'(x_{l,i}) - h_l^{-1} u(x_{l,i}) + h_l^{-1} u(x_{l,i-1}) \end{aligned}$$

$$B = -u'(x_{l,i}) + h_l^{-1} u(x_{l,i+1}) + h_l^{-1} u(x_{l,i}) \quad \text{(similar computation)}$$

$$\begin{aligned} v_{l,i} &= -\frac{h_l}{2} (A + B) \\ &= -\frac{h_l}{2} (-2h_l^{-1} u(x_{l,i}) + h_l^{-1} u(x_{l,i-1}) + h_l^{-1} u(x_{l,i+1})) \\ &= u(x_{l,i}) - \frac{1}{2} (u(x_{l,i-1}) + u(x_{l,i+1})) \end{aligned}$$

