Algorithms of Scientific Computing

FFT on Real Data

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### DFT and Symmetry – Outlook

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<td>$f_n = f_{-n}$ → Discrete Cosine Transform (DCT)</td>
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Real-valued DFT (RDFT)

For real-valued input data \( f_n \in \mathbb{R} \) (i.e. \( f_n^* := \overline{f_n} = f_n \)):

\[
F_k = \frac{1}{N} \sum_{n=-\frac{N}{2} + 1}^{\frac{N}{2}} f_n e^{-i2\pi nk/N} = \frac{1}{N} \sum_{n=-\frac{N}{2} + 1}^{\frac{N}{2}} f_n \left( \cos \left( \frac{2\pi nk}{N} \right) - i \sin \left( \frac{2\pi nk}{N} \right) \right).
\]

Properties:

1. \( \text{Re} \{ F_k \} = \frac{1}{N} \sum_{n=-\frac{N}{2} + 1}^{\frac{N}{2}} f_n \cos \left( \frac{2\pi nk}{N} \right), \)

2. \( \text{Im} \{ F_k \} = -\frac{1}{N} \sum_{n=-\frac{N}{2} + 1}^{\frac{N}{2}} f_n \sin \left( \frac{2\pi nk}{N} \right) \)

- only \( N \) independent, real coefficients necessary, since:

\[
F_k^* = \frac{1}{N} \sum f_n^* \left\{ \omega_{-nk} \right\}^* = \frac{1}{N} \sum f_n \omega_{-n(-k)} = F_{-k}
\]
Real DFT (RDFT) (2)

Map $N$ values to $N$ coefficients (and vice versa):

$$\left\{ f_{-\frac{N}{2}+1}, \ldots, f_0, \ldots, f_{\frac{N}{2}} \right\}$$

DFT $\downarrow \uparrow$ IDFT

$$\left\{ F_0, \text{Re}\{F_1\}, \text{Im}\{F_1\}, \ldots, \text{Re}\{F_{\frac{N}{2}-1}\}, \text{Im}\{F_{\frac{N}{2}-1}\}, F_{\frac{N}{2}} \right\}$$
Real DFT (RDFT) (3)

Situation:
- only \( N \) real-valued input data (as all \( N \) imaginary parts are 0)
- only \( N \) independent, real-valued coefficients due to symmetry

\[
F_{-k} = F_k^*
\]

**Wanted:** new transformation:

\( N \) real-valued input data \( \rightarrow \) \( N \) real-valued coefficients

Hence: Insert symmetry \( F_{-k} = F_k^* \) in IDFT!
Real DFT (RDFT) (4)

“Real discrete Fourier transform”:

- formulation 1

\[ F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \left( \cos \left( \frac{2\pi nk}{N} \right) - i \sin \left( \frac{2\pi nk}{N} \right) \right), \quad k = 0, \ldots, \frac{N}{2} \]

- formulation 2

\[
\begin{align*}
\text{Re}\{F_k\} &= \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \cos \left( \frac{2\pi nk}{N} \right), \quad k = 0, \ldots, \frac{N}{2} \\
\text{Im}\{F_k\} &= -\frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \sin \left( \frac{2\pi nk}{N} \right), \quad k = 1, \ldots, \frac{N}{2} - 1
\end{align*}
\]
Inverse Real DFT

We start with an input vector (Fourier coefficients) of length $2N$; then:

$$f_n = \sum_{k=-N+1}^{N} F_k e^{i2\pi nk/2N}$$

$$= F_0 + \sum_{k=1}^{N-1} \left( F_k e^{i2\pi nk/2N} + F_k e^{-i2\pi nk/2N} \right) + F_N e^{i2\pi nN/2N}$$

$$= F_0 + \sum_{k=1}^{N-1} \left( F_k e^{i2\pi nk/2N} + \left\{ F_k e^{i2\pi nk/2N} \right\}^* \right) + F_N e^{i\pi n}$$

$$= F_0 + 2 \sum_{k=1}^{N-1} \text{Re} \left\{ F_k e^{i2\pi nk/2N} \right\} + F_N e^{i\pi n}$$

$$= F_0 + 2 \sum_{k=1}^{N-1} \left( \text{Re}\{F_k\} \cos \left( \frac{\pi nk}{N} \right) - \text{Im}\{F_k\} \sin \left( \frac{\pi nk}{N} \right) \right) + F_N \cos (\pi n)$$
Inverse Real DFT (2)

Set \( a_k := 2 \text{Re}\{F_k\} \) and \( b_k := -2 \text{Im}\{F_k\} \) (but \( a_0 := \text{Re}\{F_0\} \) and \( a_N := \text{Re}\{F_N\} \)) to get:

\[
f_n = a_0 + \sum_{k=1}^{N-1} \left( a_k \cos \left( \frac{\pi nk}{N} \right) + b_k \sin \left( \frac{\pi nk}{N} \right) \right) + a_N \cos (\pi n)
\]

“Real inverse discrete Fourier transform”

Using the (real-valued) formula for \( F_k \):

\[
a_k = \frac{1}{N} \sum_{n=-N+1}^{N} f_n \cos \left( \frac{\pi nk}{N} \right), \quad b_k = \frac{1}{N} \sum_{n=-N+1}^{N} f_n \sin \left( \frac{\pi nk}{N} \right)
\]
Real-valued Trigonometric Interpolation

Interpretation of the real DFT as an interpolation problem:

- \( 2N \) ansatz functions:
  \[
  g_k(x) := \cos(kx) \quad k = 0, \ldots, N \\
  h_k(x) := \sin(kx) \quad k = 1, \ldots, N - 1
  \]

- \( 2N \) supporting points:
  \[
  x_n := \frac{2\pi n}{2N} = \frac{\pi n}{N} \quad n = -N + 1, \ldots, N
  \]

- \( 2N \) interpolation conditions:
  \[
  f_n = a_0 + \sum_{k=1}^{N-1} \left( a_k \cos \left( \frac{\pi nk}{N} \right) + b_k \sin \left( \frac{\pi nk}{N} \right) \right) + a_N \cos (\pi n)
  \]
Fast Real DFT

Computation of a real-valued DFT using complex FFT is inefficient: $N$ redundant components (symmetry)

Possibilities to improve the efficiency:

- compute two real DFTs from one complex FFT
- compute a real DFT of length $2N$ from one complex FFT of length $N$
- “compact” real FFT – use symmetry of the data directly in the algorithm
Two Real DFTs from one complex FFT

**Idea:** for real-valued \( g_n \) and \( h_n \): compute DFT of \( f_n := g_n + i h_n \):

\[
F_k = \frac{1}{N} \sum_n (g_n + i h_n) \omega_N^{-nk}
\]

Comparison with coefficients \( G_k \) and \( H_k \) of the two real DFTs:

\[
G_k = \frac{1}{N} \sum_n g_n \omega_N^{-nk} \quad H_k = \frac{1}{N} \sum_n h_n \omega_N^{-nk}
\]

Due to linearity of the Fourier transform:

\[
F_k = G_k + i H_k
\]
Two Real DFTs from one complex FFT (2)

Since $g_n$ and $h_n$ are real-valued data, the following symmetry holds:

$$G_k = G_{-k}^* \quad H_k = H_{-k}^* .$$

Hence, for $F_{-k}^*$ follows

$$F_{-k}^* = (G_{-k} + iH_{-k})^* = (G_{-k}^* + i^*H_{-k}^*) = G_k - iH_k .$$

Together with $F_k = G_k + iH_k$, we obtain

$$G_k = \frac{1}{2} \left( F_k + F_{-k}^* \right) \quad \text{and} \quad H_k = -\frac{i}{2} \left( F_k - F_{-k}^* \right) .$$
Two real DFTs from one complex FFT – Algorithm

Algorithm to compute two real DFTs:

1. set \( f_n := g_n + ih_n \)
2. compute \( F_k \) from FFT (using a library, e.g.)
3. compute \( G_k \) and \( H_k \) according to

\[
G_k = \frac{1}{2} \left( F_k + F_{-k}^* \right) \quad \text{and} \quad H_k = -\frac{i}{2} \left( F_k - F_{-k}^* \right)
\]

⇒ “half” the costs compared to using complex FFT

\textbf{but}: additional operations for pre- and postprocessing
Real DFT of length $2N$ from complex FFT of length $N$

Compute DFT of a real-valued vector $(f_{-N+1}, \ldots, f_N)$:

$$F_k = \frac{1}{2N} \sum_{-N+1}^{N} f_n \omega_{2N}^{-nk} \quad \text{for} \quad k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$$

Split up in $g_n := f_{2n}$ and $h_n := f_{2n-1}$; leads to butterfly scheme:

$$F_k = \frac{1}{2} \left( G_k + \omega_{2N}^k H_k \right) ,$$

$$F_{k \pm N} = \frac{1}{2} \left( G_k - \omega_{2N}^k H_k \right)$$

for $k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$, respectively.
Real $2N$-DFT from complex $N$-FFT

Compute $G_k$ and $H_k$ from complex FFT applied to $z_n := g_n + ih_n = f_{2n} + if_{2n-1}$:

$$G_k = \frac{1}{2} (Z_k + Z^*_{-k}) \quad \text{and} \quad H_k = -\frac{i}{2} (Z_k - Z^*_{-k})$$

Combine both schemes to:

$$F_k = \frac{1}{4} Z_k \left(1 - i\omega_{2N}^k\right) + \frac{1}{4} Z^*_{-k} \left(1 + i\omega_{2N}^k\right), \quad k = 0, \ldots, \frac{N}{2}$$

$$F_{k+N} = \frac{1}{4} Z_k \left(1 + i\omega_{2N}^k\right) + \frac{1}{4} Z^*_{-k} \left(1 - i\omega_{2N}^k\right), \quad k = -\frac{N}{2} + 1, \ldots, 0$$
Algorithm for a real $2N$-DFT:

(1) set $z_n := f_{2n} + if_{2n-1}$

(2) compute $Z_k$ from FFT applied on $z_n$ (using a library, e.g.)

(3) compute $F_k$ according to

\[
F_k = \frac{1}{4} Z_k \left( 1 - i\omega_{2N}^k \right) + \frac{1}{4} Z_{-k}^* \left( 1 + i\omega_{2N}^k \right), \quad k = 0, \ldots, \frac{N}{2}
\]

\[
F_{k+N} = \frac{1}{4} Z_k \left( 1 + i\omega_{2N}^k \right) + \frac{1}{4} Z_{-k}^* \left( 1 - i\omega_{2N}^k \right), \quad k = -\frac{N}{2} + 1, \ldots, 0
\]

⇒ Complexity determined by complex $N$-FFT

plus: additional operations for pre- and postprocessing
Compact Real FFT

Compute DFT of a real-valued vector \((f_{-N+1}, \ldots, f_N)\):

\[
F_k = \frac{1}{2N} \sum_{-N+1}^{N} f_n \omega_{2N}^{-nk} \quad \text{for } k = 0, \ldots, N
\]

Split up in \(g_n := f_{2n}\) and \(h_n := f_{2n-1}\); leads to butterfly scheme:

\[
F_k = \frac{1}{2} \left( G_k + \omega_{2N}^k H_k \right) \quad \text{for } k = 0, \ldots, \frac{N}{2},
\]

\[
F_{k+N} = \frac{1}{2} \left( G_k - \omega_{2N}^k H_k \right) \quad \text{for } k = -\frac{N}{2} + 1, \ldots, 0.
\]

\(G_k\) and \(H_k\) are coefficients of a real-valued DFT of length \(N\); hence:

\[
G_k = G_{-k}^* \quad \text{and} \quad H_k = H_{-k}^* \quad \text{for } k = 0, \ldots, \frac{N}{2} - 1
\]
Compact Real-valued FFT (2)

Use symmetry of $G_k$ and $H_k$ for the computation of $F_k$:

$$F_k = \frac{1}{2} \left( G_k + \omega_{2N}^k H_k \right) \quad \text{für } k = 0, \ldots, \frac{N}{2},$$

$$F_{N-k} = \frac{1}{2} \left( G_{-k} - \omega_{2N}^{-k} H_{-k} \right)$$

$$= \frac{1}{2} \left( G_k - \omega_{2N}^k H_k \right)^* \quad \text{for } k = 0, \ldots, \frac{N}{2} - 1$$

⇒ Computation of $F_k$ (for $k = 0, \ldots, N$) reduced to the computation of $G_k$ and $H_k$ (for $k = 0, \ldots, \frac{N}{2}$, respectively).

“Edson’s algorithm” (1968)