

Algorithms of Scientific Computing

Discrete Cosine Transformation

Exercise 1: Two-dimensional Cosine Transformation

The JPEG-method computes the coefficients \tilde{F}_{kl} from the image data f_{nm} using the following formula

$$\tilde{F}_{kl} = \frac{1}{N \cdot M} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{nm} \cos\left(\frac{\pi k (n + \frac{1}{2})}{N}\right) \cos\left(\frac{\pi l (m + \frac{1}{2})}{M}\right)$$

Assume you have a procedure that can compute all coefficients \tilde{G}_k , $k = 0, \dots, N-1$ efficiently, according to the formula

$$\tilde{G}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cos\left(\frac{\pi k (n + \frac{1}{2})}{N}\right).$$

How can you compute the coefficients \tilde{F}_{kl} using this procedure?

Exercise 2: Discrete Cosine Transformation

We start with a dataset f_{-N+1}, \dots, f_N , which fulfills the following symmetry constraint:

$$f_{-n} = f_n \quad \text{for } n = 1, \dots, N-1$$

a) Show that the corresponding Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^N f_n \omega_{2N}^{-kn} \tag{1}$$

are real values only and can be written as:

$$F_k = \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi nk}{N}\right) + \frac{1}{2} f_N \cos(\pi k) \right). \tag{2}$$

b) Show that the F_k is symmetric too.

- c) Let $\text{FFT}(\mathbf{f}, N)$ be a procedure that computes the coefficients F_k efficiently (according to equation (1)) from a field \mathbf{f} which consists of $2N$ values f_n . (The result is written back to field \mathbf{f})

Write a short procedure $\text{DCT}(\mathbf{g}, N)$ which uses procedure FFT to compute the coefficients F_k for $k = 0, \dots, N$ from equation (2) for the (non-symmetrical) data f_0, \dots, f_N , stored in the parameter field \mathbf{g} .

Exercise 3: Fast Discrete Cosine Transformation

Formulate the butterfly scheme for equation (1) from the previous exercise. Divide the dataset f_n of length $2N$ into a dataset $g_n := f_{2n}$, containing all values with an even index, and a dataset $h_n := f_{2n-1}$, with all values with odd index. Which symmetries can be found in g_n and h_n ? Of which kind (Cosine/Sine Transformation, DFT with real data) are the according DFTs of length N ? Which symmetries can be found if the dataset f_n fulfills the following symmetry constraint:

$$f_{-n} = f_{n+1}.$$