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Sheet 8
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Adaptive Sparse Grids, Orthogonality

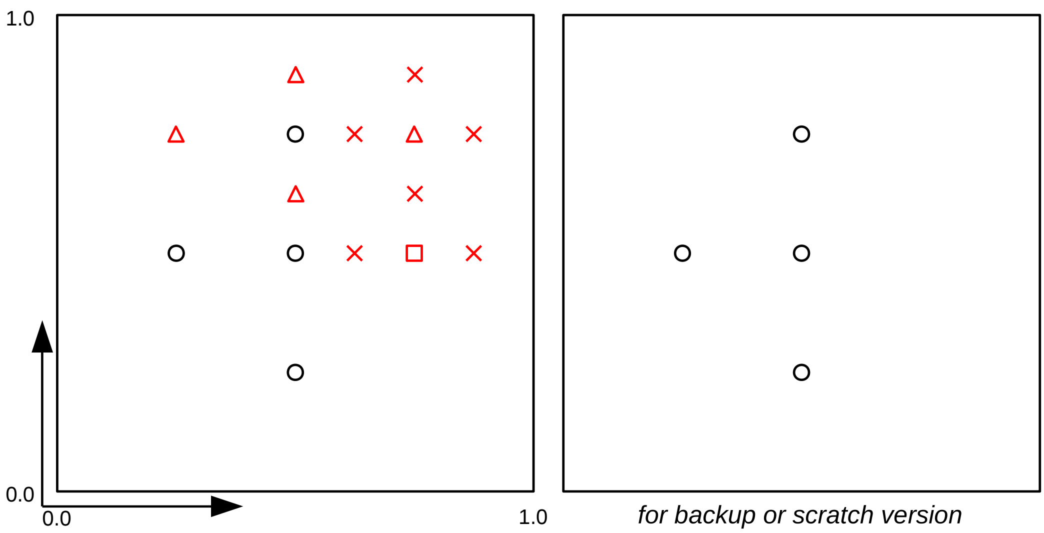
Proposed solution

1 Adaptive Sparse Grids

Here, the exercise is to adaptively refine a 2-dimensional sparse grid without boundary. We follow the notation introduced in the lecture and also choose our domain accordingly with $\Omega = [0.0, 1.0]^2$.

1. In the following image you see an incomplete regular sparse grid V_2^1. Insert the missing grid points using small **squares**. What are the level-index-vector pairs $\vec{l}, \vec{i}$ for each of them?

   $$(2,1),(3,1)$$



2. Use the (modified) picture from the previous task to perform two steps of adaptive refinement:

   (a) Refine grid point $\vec{l}, \vec{i} = (1,2),(1,3)$: create all hierarchical children. Draw its children as small **triangles**. Make sure that you also insert all missing hierarchical parents (and parents of parents, ...) of these children to make the grid suitable for typical algorithms on sparse grids.

(b) Now refine grid point $(2,2),(3,3)$. Again, do not forget to create all missing parents. Draw all new points as small **crosses**.

2 Orthogonality

We first consider orthogonality of functions $[a,b] \to \mathbb{R}$ in the two scalar products we already know

- L^2 scalar product:
$$(u,v)_2 := \int_a^b u(x)v(x)\ dx$$
- "energy scalar product":
$$(u,v)_a := \int_a^b u'(x)v'(x)\ dx,$$

We assume that the space of functions under consideration again be well-defined such that $(u,u) > 0$ for $u \neq 0$ is ensured.

(i) Show that for $g_k : [0,2\pi] \to \mathbb{R}, g_k(x) = \sin(kx)$ and $k,j \in \mathbb{N}$ the L_2 scalar product is

$$(g_k, g_j)_2 = \begin{cases} 0 & \text{for } k \neq j, \\ \pi & \text{else.} \end{cases}$$

$$\begin{aligned} \int_0^{2\pi} \sin(kx)\sin(jx)\ dx &= \underbrace{-\frac{1}{k}\cos(kx)\sin(jx)\Big|_0^{2\pi}}_{=0} + \frac{j}{k}\int_0^{2\pi} \cos(kx)\cos(jx)\ dx \\ &= \underbrace{\frac{j}{k^2}\sin(kx)\cos(jx)\Big|_0^{2\pi}}_{=0} + \frac{j^2}{k^2}\int_0^{2\pi} \sin(kx)\sin(jx)\ dx \end{aligned}$$

This only holds for $j \neq k$ iff $(g_k, g_j)_2 = 0$

Recalling the results from worksheet 5 (back then the integral's domain was $[0,1]$) we know $(g_k, g_k) = \|g_k\|_2^2 = \pi$.

(ii) Which functions of the hierarchical basis are orthogonal to each other w.r.t. the L_2 scalar product? What about the energy scalar product?
Because of $\phi_{l,i}(x) \geq 0$ we have $(\phi_{l,i}, \phi_{l',j})_2 = 0$ iff the supports are disjoint (i.e. there's no ancestor relation between them in the binary tree).

For arbitrary pairs however we get for the energy scalar product $(\phi_{l,i}, \phi_{l',j})_a = 0$ (draw derivatives and think about how basis functions influence each other).
A direct implication of this is that in the one-dimensional case the stiffness matrix containing the energy scalar product of the hierarchical basis functions is a diagonal matrix!

(iii) Let V be a vector space with $\dim V = n < \infty$ with scalar product $(. , .)$ and associated norm $\|x\| := \sqrt{(x,x)}$. Also let $\Psi = \{\psi_1, \ldots, \psi_n\} \subset V$ a orthonormal system, i.e.

$$(\psi_i, \psi_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{else.} \end{cases}$$

a) Show that for

$$x = \sum_{i=1}^{n} \alpha_i \psi_i$$

the following holds:

$$\|x\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

$$\|x\|^2 \quad = \quad (x,x) = \left(\sum_{i=1}^{n} \alpha_i \psi_i, \sum_{j=1}^{n} \alpha_j \psi_j \right) = \sum_{i,j=1}^{n} (\alpha_i \psi_i, \alpha_j \psi_j) = \sum_{i=1}^{n} \alpha_i^2$$

Now **assume** *for a moment that the hierarchical basis was an orthonormal basis: What would the error estimation for the error $\|u - u_L\|$ look like?*

$$\|u - u_L\| \leq \sum_{\vec{l} \notin L} \|w_{\vec{l}}\|$$

could be rewritten as an exact equation

$$\|u - u_L\| = \sum_{\vec{l} \notin L} \sum_{\vec{i} \in I_{\vec{l}}} (v_{\vec{l},\vec{i}})^2$$

b) Show that Ψ is a linearly independent system!

If $0 = \sum \alpha_i \psi_i$ then with the previous item we get $\alpha_1 = \ldots = \alpha_n = 0$.

c) Show for every $x \in V$:

$$x = \sum_{i=1}^{n} (x, \psi_i) \psi_i.$$

Since we have n linearly independent ψ_i there's always a well-defined set of unique α_i with

$$x = \sum_{i=1}^{n} \alpha_i \psi_i.$$

Therefore we have

$$(x, \psi_j) = \left(\sum_{i=1}^{n} \alpha_i \psi_i, \psi_j \right) = \alpha_j.$$