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Algorithms of Scientific Computing

– Integral representation of the hierarchical surplus –

Theorem:

For a function u with bounded second derivative, we can write its hierarchical surpluses as:

$$v_{l,i} := u(x_{l,i}) - \frac{1}{2}(u(x_{l,i-1}) + u(x_{l,i+1})) = \int_\Omega \psi_{l,i}(x) u''(x) dx; \qquad \psi_{l,i}(x) = -\frac{h_l}{2}\phi_{l,i}(x)$$

Proof:

$$\begin{aligned}
v_{l,i} &\stackrel{?}{=} \int_\Omega -\frac{h_l}{2}\phi_{l,i}(x)u''(x)dx \\
&= -\frac{h_l}{2}(\underbrace{\int_{x_{l,i-1}}^{x_{l,i}} \phi_{l,i}(x)u''(x)dx}_{=:\,A} + \underbrace{\int_{x_{l,i}}^{x_{l,i+1}} \phi_{l,i}(x)u''(x)dx)}_{=:\,B}
\end{aligned}$$

(Note: $\phi_{l,i}(x)$ a linear function in $[x_{l,i-1}, x_{l,i}]$ and $[x_{l,i}, x_{l,i+1}]$, resp.)

$$\begin{aligned}
A &= [\phi_{l,i}(x)u'(x)]_{x_{l,i-1}}^{x_{l,i}} - \int_{x_{l,i-1}}^{x_{l,i}} \phi'_{l,i}(x)u'(x)dx && \text{(integration by parts)} \\
&= u'(x_{l,i}) - h_l^{-1}\int_{x_{l,i-1}}^{x_{l,i}} u'(x)dx && \text{(as } \phi'_{l,i}(x) = h_l^{-1}) \\
&= u'(x_{l,i}) - h_l^{-1}u(x_{l,i}) + h_l^{-1}u(x_{l,i-1})
\end{aligned}$$

$$B = -u'(x_{l,i}) + h_l^{-1}u(x_{l,i+1}) + h_l^{-1}u(x_{l,i}) \qquad \text{(similar computation)}$$

$$\begin{aligned}
v_{l,i} &= -\frac{h_l}{2}(A+B) \\
&= -\frac{h_l}{2}\left(-2h_l^{-1}u(x_{l,i}) + h_l^{-1}u(x_{l,i-1}) + h_l^{-1}u(x_{l,i+1})\right) \\
&= u(x_{l,i}) - \frac{1}{2}\big(u(x_{l,i-1}) + u(x_{l,i+1})\big)
\end{aligned}$$

□