

Algorithms of Scientific Computing

Hierarchical Methods and Sparse Grids

Michael Bader

Technische Universität München

Summer Term 2013

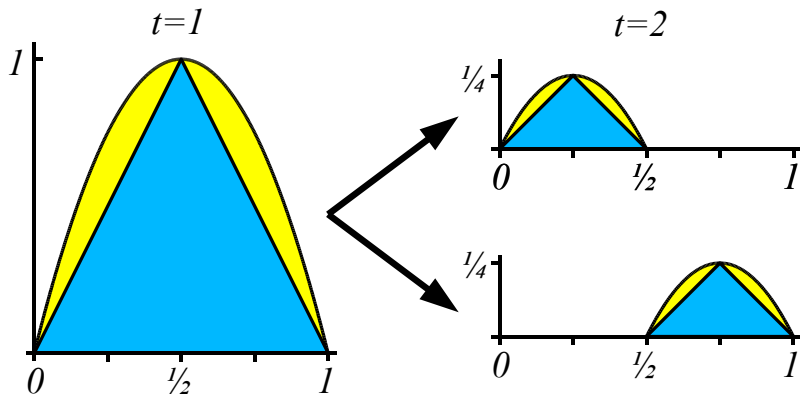


Part II

Hierarchical Decomposition, 1d

Archimedes' Quadrature

Compute an approximation of $F_1 := \int_0^1 4 \cdot x \cdot (1-x) dx = \frac{2}{3}$



Archimedes' Quadrature (2)

- Integrating $4x(1-x)$, we have to consider several quantities
- Ordered by (recursive) level t :

Level-depth	1	2	3	4	...	t
Mesh-width h	$1/2$	$1/4$	$1/8$	$1/16$...	2^{-t}
# triangles	1	2	4	8	...	$\frac{1}{2}2^t$
surplus v	1	$1/4$	$1/16$	$1/64$...	$4 \cdot 2^{-2t}$
Area of triangle D_1	$1/2$	$1/16$	$1/128$	$1/1024$...	$4 \cdot 2^{-3t}$
Sum (current t)	$1/2$	$1/8$	$1/32$	$1/128$...	$2 \cdot 2^{-2t}$
Sum ($\leq t$)	$1/2$	$5/8$	$21/32$	$85/128$...	$\frac{2}{3}(1 - 2^{-2t})$
Error	$1/6$	$1/24$	$1/96$	$1/384$...	$\frac{2}{3}2^{-2t}$

Approximation of Functions

- To analyze Archimedes' quadrature rule, we consider functions
- We need a representation of the (approximating) function $u(x)$ which we are integrating:
 - u as linear combination of ansatz functions ϕ_j :

$$u(x) = \sum_{i=1}^n \alpha_i \cdot \phi_i(x)$$

- Integrating $u(x)$:

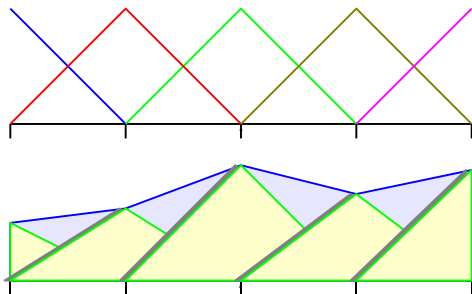
$$\int_a^b u(x) dx = \sum_i^n \alpha_i \int_a^b \phi_i(x) dx,$$

- Weighted sum of α_j
- Remember: Newton-Cotes formulas are weighted sum of function evaluations

Composite Trapezoidal Rule: Function

Interpolant

- Continuous, piecewise linear function
- Represent u in nodal point (hat) basis



- Coefficients α_j are function values at grid points
- Ansatz functions have area h ($h/2$ at boundaries)

Piecewise Linear Functions

Ansatz space

- Only consider $u : [0, 1] \rightarrow \mathbb{R}$
- Consider discretization level $n \in \mathbb{N}$
- Mesh-width $h_n = 2^{-n}$
- Grid points $x_{n,i} = i \cdot h_n$
- Define “mother of all hat functions”

$$\phi(x) := \max\{1 - |x|, 0\}$$

⇒ Ansatz functions

$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- Nodal point basis $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$

Piecewise Linear Functions (2)

- Space of continuous piecewise linear functions

$$V_n = \text{span}(\Phi_n)$$

- Interpolants $u_n \in V_n$

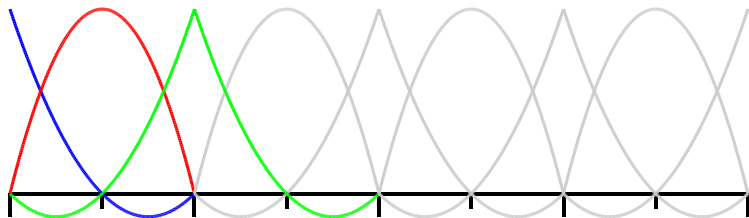
$$u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i} \phi_{n,i}(x)$$

- V_n the space of all such interpolants u_n

Composite Simpson's Rule: Function

Interpolant

- Continuous, piecewise quadratic function
- More complicated basis:



- Ansatz functions: Lagrangian polynomials, glued together
- α_j : function values at grid points
- Ansatz functions have area $h/6$ (blue), $4h/6$ (red), $2h/6$ (green)
- We'll not formally define basis functions here...

From Composite Trapezoidal to Archimedes

Piecewise linear functions

- We restrict our functions u to $u(0) = u(1) = 0$
- Nodal point basis for discretization level n :

$$\Phi_n := \{\phi_{n,i}, 1 \leq i \leq 2^n - 1\}$$

- Wanted: *function space*

$$V := \bigcup_{l=1}^{\infty} V_l$$

contains all functions which are in V_l for sufficiently large l

- However: generating system of V as

$$\Phi := \bigcup_{l=1}^{\infty} \Phi_l$$

does not lead to a basis (not linear independent)

Hierarchical Basis

- We are interested in a hierarchical decomposition of V_l
- Define *hierarchical increment* W_l , such that V_l is a *direct sum*:

$$V_l = V_{l-1} \oplus W_l$$

Side-note: direct sum

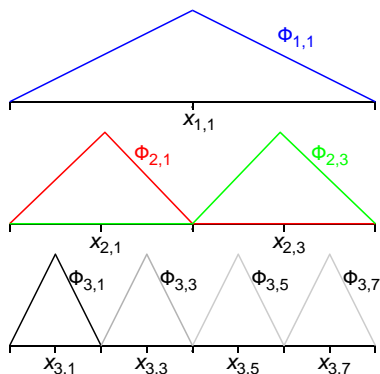
- Every $u_l \in V_l$ can be uniquely decomposed as $u_l = u_{l-1} + w_l$, with $u_{l-1} \in V_{l-1}$ and $w_l \in W_l$
- W_l has to contain 2^{l-1} ansatz functions:
 $\dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l$
- This holds (introducing index sets \mathcal{I}_l) for

$$\mathcal{I}_l := \{i : 1 \leq i < 2^l, i \text{ odd}\}$$

$$W_l := \text{span} \{\phi_{l,i} : i \in \mathcal{I}_l\}$$

Hierarchical Increments

- Set of hierarchical increments W_l
- For $l = 1$: $W_1 = V_1$
- Example for $l = 1, 2, 3$:



Hierarchical Basis (cont.)

- Then

$$V_n = \bigoplus_{l=1}^n W_l$$

is a direct sum, too:

- $u \in V_n$ can be decomposed uniquely into $w_l \in W_l$:

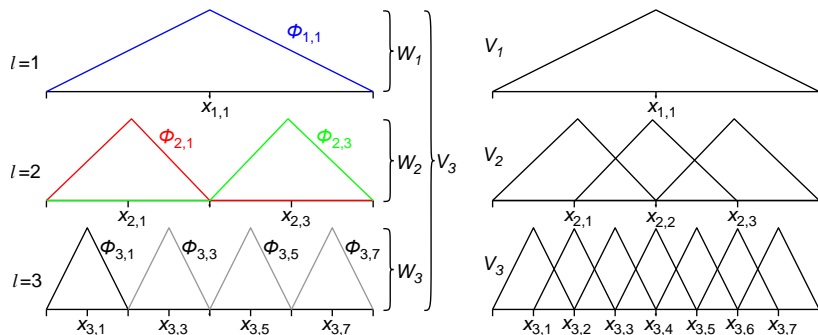
$$u = \sum_{l=1}^n w_l = \sum_{l=1}^n \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i}$$

→ Coefficients $v_{l,i}$ are hierarchical surplusses

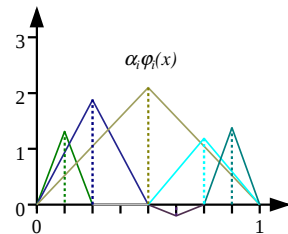
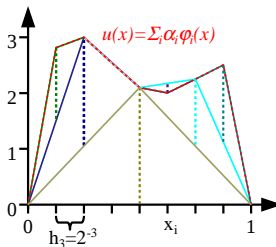
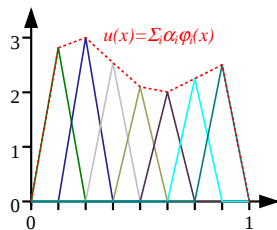
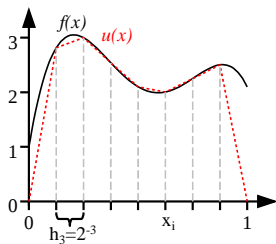
- Corresponding basis of V_n (or, with ∞ instead of n , of V)

$$\Psi_n := \bigcup_{l=1}^n \{\phi_{l,i} : i \in \mathcal{I}_l\}.$$

Comparison



Comparison (2)



Analysis of Hierarchical Decomposition

- Contribution of summands in hierarchical decomposition

$$u = \sum_{l=1}^n w_l = \sum_{l=1}^n \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i}.$$

- Interesting in univariate setting
- Will be crucial in multivariate setting
→ Cost/benefit analysis will help to significantly reduce effort
- Need several norms to measure w_l (cf. worksheet 5)

Norms of Functions

Again, we assume sufficiently smooth functions $u : [0, 1] \rightarrow \mathbb{R}$

Norms

- Maximum-norm

$$\|u\|_{\infty} := \max_{x \in [0, 1]} |u(x)|$$

- L^2 -norm

$$\|u\|_2 := \sqrt{\int_0^1 u(x)^2 dx},$$

for the L^2 scalar product

$$(u, v)_2 := \int_0^1 u(x)v(x) dx$$

- Energy-norm

$$\|u\|_E := \|u'\|_2$$

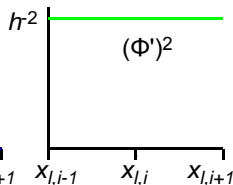
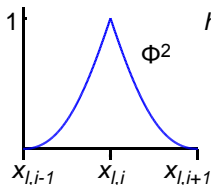
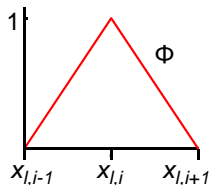
Norms of Basis Functions

For the basis functions $\phi_{l,i}$, we obtain

$$\|\phi_{l,i}\|_{\infty} = 1$$

$$\|\phi_{l,i}\|_2 = \sqrt{\frac{2h_l}{3}}$$

$$\|\phi_{l,i}\|_E = \sqrt{\frac{2}{h_l}}$$



Estimation of Surpluses

- Let $\psi_{l,i} := -\frac{h_l}{2} \phi_{l,i}$
 - Surplus $v_{l,i}$ of basis function $\phi_{l,i}$
 - u two times differentiable
- ⇒ We can then write $v_{l,i}$ as (see separate proof)

$$v_{l,i} = \int_0^1 \psi_{l,i}(x) u''(x) dx.$$

- $v_{l,i}$ depends on u'' , thus we define for future use

$$\mu_2(u) := \|u''\|_2 \quad \text{and} \quad \mu_\infty(u) := \|u''\|_\infty.$$

→ note: $\mu_2(u)$ and $\mu_\infty(u)$ are properties of the function u

Estimation of Surplusses (2)

- Starting from integral representation of $v_{l,i}$, we can bound

$$|v_{l,i}| \leq \frac{h_l}{2} \cdot \left(\int_0^1 \phi_{l,i} dx \right) \cdot \mu_\infty(u) = \frac{h_l^2}{2} \cdot \mu_\infty(u)$$

- and, via Cauchy-Schwartz inequality $|(u, v)| \leq \|u\| \cdot \|v\|$,

$$|v_{l,i}| \leq \frac{h_l}{2} \|\phi_{l,i}\|_2 \cdot \mu_2(u|_{T_i}) = \sqrt{\frac{h_l^3}{6}} \cdot \mu_2(u|_{T_i}),$$

where $u|_{T_i}$ restricts u to the support $T_i = [x_{l,i-1}, x_{l,i+1}]$ of $\phi_{l,i}$

Estimation of w_l

- Estimate contribution of level l , i.e.

$$w_l = \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i}$$

in hierarchical decomposition of u

- Use that supports of $\phi_{l,i}$ are pairwise disjoint
- Maximum-norm

$$\|w_l\|_\infty \leq \frac{h_l^2}{2} \cdot \mu_\infty(u),$$

- L^2 -norm

$$\|w_l\|_2^2 = \sum_{i \in \mathcal{I}_l} |v_{l,i}|^2 \cdot \|\phi_{l,i}\|_2^2 \leq \frac{h_l^3}{6} \cdot \frac{2h_l}{3} \cdot \sum_{i \in \mathcal{I}_l} \mu_2(u|_{T_i})^2 = \frac{h_l^4}{9} \mu_2(u)^2,$$

$$\Rightarrow \|w_l\|_2 \in \mathcal{O}(h_l^2)$$

Estimation of w_l (2)

- Energy-norm

$$\begin{aligned}\|w_l\|_E^2 &= \sum_{i \in \mathcal{I}_l} |v_{l,i}|^2 \cdot \|\phi_{l,i}\|_E^2 = \sum_{i \in \mathcal{I}_l} |v_{l,i}|^2 \frac{2}{h_l} \\ &\leq \frac{2}{h_l} \cdot \frac{h_l^4}{4} \cdot \frac{1}{2h_l} \mu_\infty(u)^2 = \frac{h_l^2}{4} \mu_\infty(u)^2\end{aligned}$$

($2^{l-1} = 1/(2h_l)$ summands)

$\Rightarrow \|w_l\|_E \in \mathcal{O}(h_l)$

Estimation of w_l (3)

- We can write u (twice differentiable) as infinite series

$$u = \sum_{l=1}^{\infty} w_l$$

- Convergent in all three norms
- With

$$u - u_n := u - \sum_{l=1}^n w_l = \sum_{l=n+1}^{\infty} w_l$$

in maximum- and L^2 -norm $\mathcal{O}(h_n^2)$, in energy-norm $\mathcal{O}(h_n)$