Algorithms of Scientific Computing

Hierarchical Methods and Sparse Grids

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Part VII

Algorithms and Data Structures for Sparse Grids

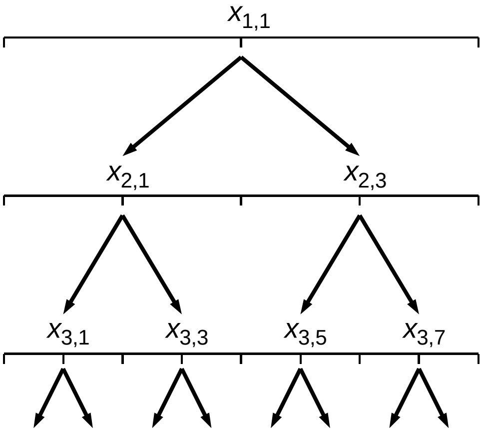
Algorithms and Data Structures

- We will now look at typical sparse grid algorithms
- Can, e.g., be used for hierarchization/dehierarchization, integration, data mining, solution of PDE, ...
- important: *adaptive* representation
- Algorithms depend on data structure:
  - Efficient traversal of sparse grid necessary
  - Thus, we deal with data structures for sparse grids, too

Data Structures ($d = 1$)

- How to store function $u : [0, 1] \to \mathbb{R}$ in hierarchical representation (i.e. surplusses $v_{\vec{l},\vec{i}}$)?
- Order and store grid points and associated values in binary tree
  - Root is node $x_{1,1} = 1/2$
  - Children of node $x_{l,i}$ are – if existent – the grid points $x_{l+1,2i-1}$ and $x_{l+1,2i+1}$ of level $l + 1$
  - Alternative point of view if child does not exist:
    Complete subtree of binary tree starting from child with all surplusses set to 0

Data Structures ($d = 1$) (2)



Typical Algorithms ($d = 1$)

Hierarchization and Dehierarchization

- Prototype for typical algorithm (c.f. worksheet 4)
- Our data structure has to allow
  1. Iteration over all grid points, considering the hierarchical relations
     - E.g. for hierarchization: first handle all grid points in the support of $\phi_{l,i}$, then compute $v_{l,i}$
  2. Access to *hierarchical neighbors*: grid points at interval boundaries of support of $\phi_{l,i}$ (if possible – exception for points 0 and 1 as not in the tree), e.g. to compute

$$v_{l,i} = u_{l,i} - \frac{1}{2}(u_l + u_r).$$

Typical Algorithms ($d = 1$) (2)

- Hierarchical neighbors are easy to find geometrically

$$x_{l,i-1}, \qquad x_{l,i+1}$$

- But have even indices $\Rightarrow$ really are on another level ($< l$)
- In the binary tree structure:
  - Can be found on way from root to node
  - One is parent node
- For hierarchization/dehierarchization: pass hierarchical neighbors as additional parameters

- Developing algorithms:
  - Try to store all information to process one node at the node and its hierarchical neighbors
  - Access to other nodes typically expensive
  - Tree traversal with “supply of hierarchical neighbors” only linear in number of nodes

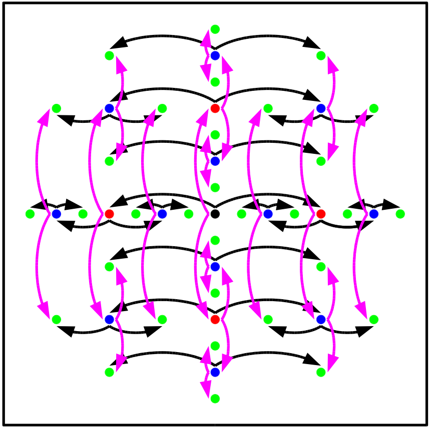
Data Structures and Typical Algorithms ($d > 1$)

- What data structure to use in more than one dimension?
- Algorithmically: use construction of basis functions as product of one-dimensional hats. Ideally:
  - Use a loop $1, \dots, d$ over the dimension
  - Apply $1d$ algorithm on one-dimensional structures in each dimension (see also worksheet 7)

$\Rightarrow$ Need access to hierarchical neighbors in each spacial direction; implies to create binary tree structure in each dimension

- Disadvantages:
  - Storage requirements ($2d$ pointers)
  - High effort to keep structure consistent when inserting or deleting points

Data Structures and Typical Algorithms ($d > 1$) (2)



If you could recognize anything, it would be binary tree structures for rows (black) and columns (magenta)

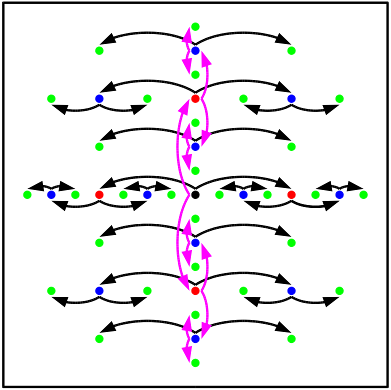
Data Structures and Typical Algorithms ($d > 1$) (3)

Often better:

- Store in a node only two pointers for one direction (e.g. x_1)
- A binary tree of nodes is a row (a $1d$ structure parallel to the x_1 axis)
- For next spacial direction x_2, only a binary tree in x_2 direction required
- Stores one plane parallel to x_1–x_2 coordinate plane; nodes are the binary trees with $1d$ structures
- For each additional spatial direction x_d build binary tree with $(d-1)$-dimensional structures as nodes

- Disadvantage: Access to hierarchical neighbors not that easy any more (except for x_1-direction)
- But can be achieved without much more computational effort by suitable reordering of loops and tree traversals

Data Structures and Typical Algorithms ($d > 1$) (4)



Already more clear: One plane (two-dimensional structure) consists of one binary tree (magenta) of which the nodes are binary trees (black) for each row

Data Structures and Typical Algorithms ($d > 1$) (5)

Hash table

- Much more comfortable (and not too inefficient) alternative
- Store coefficients as target values, with, e.g., $(\vec{l}, \vec{i})$ as keys
- No need to care about tree structures
- Only requires computation of indices of accessed nodes (hierarchical neighbor, ...)

$\Rightarrow$ Best solution for your own sparse grid experiments

Further assumptions on data structures

- Algorithms will assume that all hierarchical neighbors exist for each grid point

$\Rightarrow$ If creating grid points adaptively, create them if necessary

- No further assumptions

Summary

Data Structures

- array-based for regular sparse grids and combination technique (see tutorials)
- hierarchical adaptivity reflected by tree-based data structures (but: more complicated in higher dimensions)
- hash-based data structures

Algorithms

- hierarchisation and dehierarchisation: tree-based recursion plus "hierarchical neighbours"
- archimedes quadrature $\rightarrow$ recursion on dimensions
- much more complicated algorithms, if we want to use sparse grids for solution of partial differential equations