Algorithms of Scientific Computing

From Quadtrees to Space-Filling Curves

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Overview: Modelling of Geometric Objects

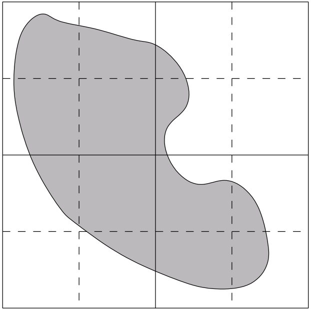
Surface-oriented models:

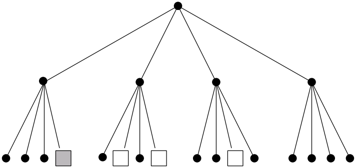
- wire-frame models
- augmented models using Bezier curves and planes
- typically described by graphs on nodes, edges, and faces

Volume-oriented models:

- Constructive Solid Geometry (boolean operations on primitives)
- voxel models: place object in a grid
- octrees: recursive refinement of voxel grids

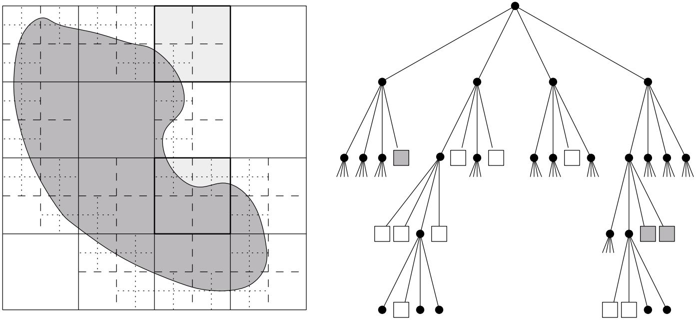
Quadtrees to Describe Geometric Objects





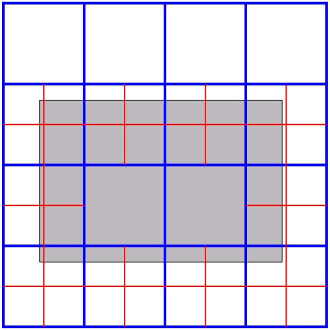
- start with an initial square (covering the entire domain)
- recursive substructuring in four subsquares

Quadtrees to Describe Geometric Objects

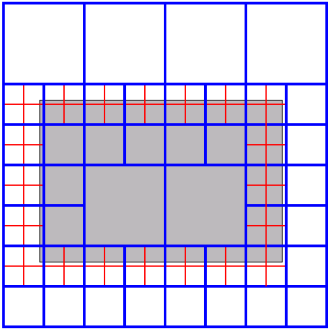


- start with an initial square (covering the entire domain)
- recursive substructuring in four subsquares
- adaptive refinement possible
- terminate, if squares entirely within or outside domain

Number of Quadtree Cells to Store a Rectangle



k=2

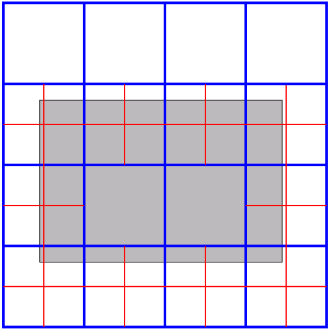


k=3

Terminal (t_k) and boundary (b_k) cells after k refinement steps:

$$\begin{aligned} b_k &= 2 \cdot b_{k-1} \\ t_k &= t_{k-1} + 2 \cdot b_{k-1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} b_k &= 2^{k-2} \cdot b_2 = \tfrac{5}{2} \cdot 2^k \\ t_k &= \ldots = 5 \cdot 2^k - 14 \end{aligned}$$

Number of Quadtree Cells to Store a Rectangle



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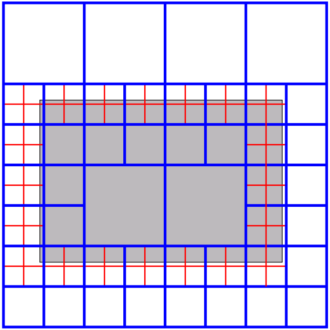
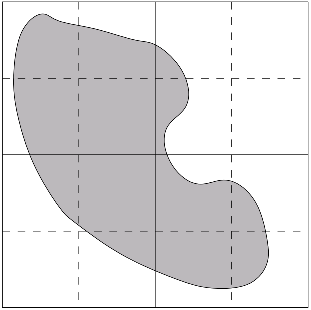


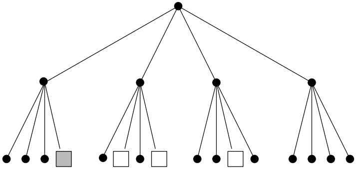
k=3

- uniformly ref. voxel-grid (level k): $\left(2^{d=2}\right)^k = \left(2^k\right)^2 =: \mathcal{O}(N^2)$ cells
- quadtree-refined grid (level k): $\frac{15}{2} \cdot 2^k - 14 =: \mathcal{O}(N)$ cells

  $\Rightarrow$ number of cells proportional to length of boundary ($N := 2^k$)

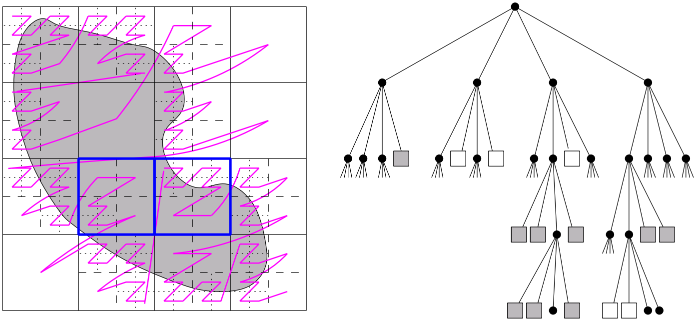
Storing a Quadtree – Sequentialisation





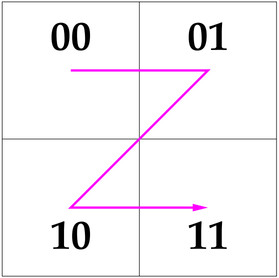
- sequentialise cell information according to **depth-first traversal**
- relative numbering of the child nodes determines sequential order

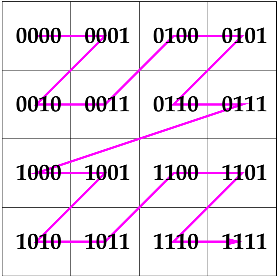
Storing a Quadtree – Sequentialisation



- sequentialise cell information according to **depth-first traversal**
- relative numbering of the child nodes determines sequential order
- here: leads to so-called **Morton order**

Morton Order (”Z curve“)





Relation to bit arithmetics:

- odd digits: position in vertical direction
- even digits: position in horizontal direction

Morton Order and Cantor's Mapping

Georg Cantor (1877):

$$0.01111001\ldots \rightarrow \begin{pmatrix} 0.0110\ldots \\ 0.1101\ldots \end{pmatrix}$$

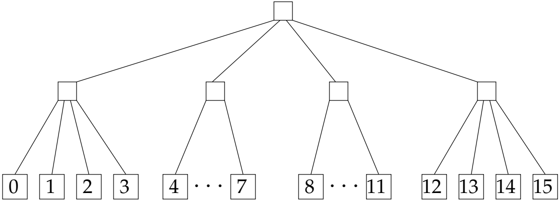
- **bijective** mapping $[0,1] \rightarrow [0,1]^2$
- proved identical cardinality of $[0,1]$ and $[0,1]^2$
- provoked the question: is there a **continuous** mapping?
  (i.e. a curve)

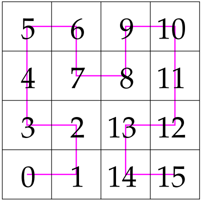
Preserving Neighbourship for a 2D Octree

Requirements:

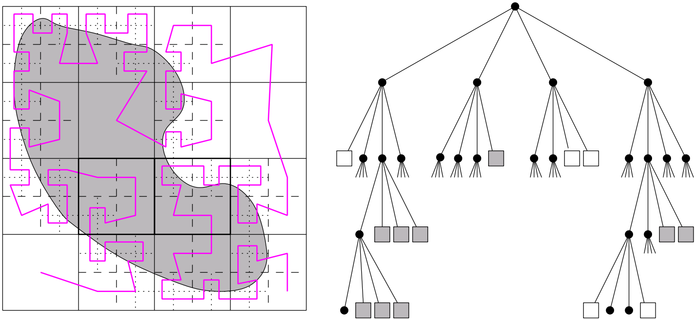
- consider a simple 4 $\times$ 4-grid
- uniformly refined
- subsequently numbered cells should be neighbours in 2D

Leads to (more or less unique) numbering of children:



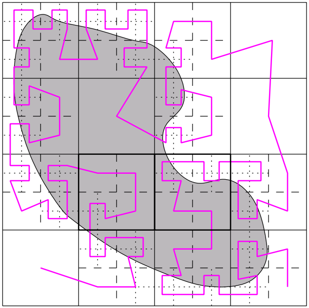


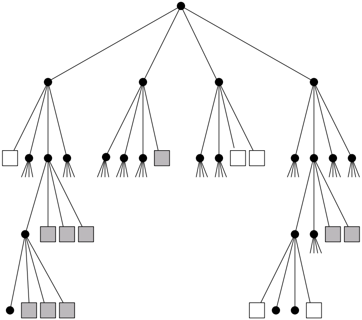
Preserving Neighbourship for a 2D Octree (2)



- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D

Preserving Neighbourship for a 2D Octree (2)





- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D
- here: similar to the concept of **Hilbert curves**

Open Questions

Algorithmics:

- How do we describe the sequential order algorithmically?
- What kind of operations are possible?
- Are there further “orderings” with the same or similar properties?

Applications:

- Can we quantify the “neighbour” property?
- In what applications can this property be useful?
- Which other properties and/or operations can be useful?