Algorithms of Scientific Computing

Space-Filling Curves and their Applications in Scientific Computing (3)

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Summer Term 2013

Approximating Polygons of the Hilbert Curve

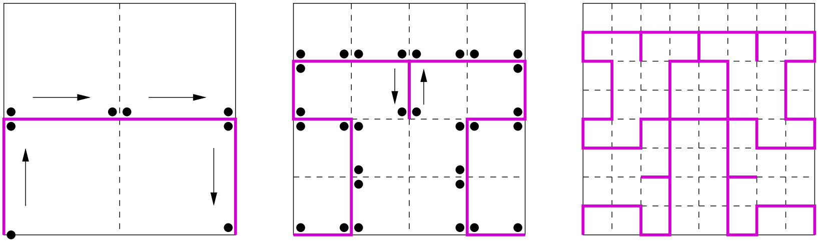
Idea: Connect start and end point of iterate on each subcell.

Definition:

The straight connection of the $4^n + 1$ points

$$h(0), h(1 \cdot 4^{-n}), h(2 \cdot 4^{-n}), \ldots, h((4^n - 1) \cdot 4^{-n}), h(1)$$

is called the *n-th approximating polygon of the Hilbert curve*



Properties of the Approximating Polygon

- the approximating Polygon connects the **corners** of the recursively divided subsquares
- the connected corners are start and end points of the space-filling curve within each subsquare

  ⇒ **assists in the construction of space-filling curves**

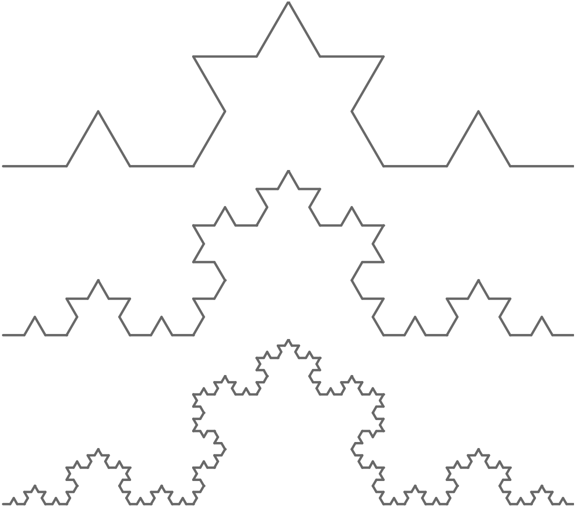
- approximating polygons are constructed by recursive repetition of a so-called **Leitmotiv**

  ⇒ **similarity to Koch and other fractal curves**

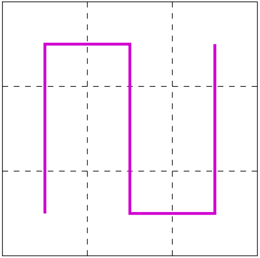
- the sequence of corresponding functions $p_n(t)$ converges **uniformly** towards h

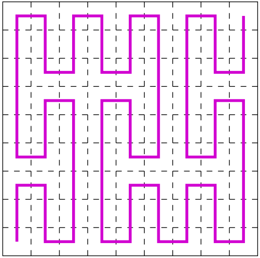
  ⇒ additional proof of continuity of the Hilbert curve

Example: Koch Curve



Construction of the Peano Curve





Recursive Construction:

- divide quadratic domain into 9 subsquares
- construct Peano curve for each subsquare
- join the partial curves to build a higher level curve

Arithmetic Formulation of the Peano Function

t given in "nonal" system, $t = 0_9.n_1 n_2 n_3 n_4 \ldots$, then

$$p(0_9.n_1 n_2 n_3 n_4 \ldots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the operators

$$P_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + 0 \\ \frac{1}{3}y + \frac{2}{3} \end{pmatrix} \qquad P_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{1}{3} \\ -\frac{1}{3}y + 1 \end{pmatrix} \qquad P_8 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y + \frac{2}{3} \end{pmatrix}$$

$$P_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + \frac{1}{3} \\ \frac{1}{3}y + \frac{1}{3} \end{pmatrix} \qquad P_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + \frac{2}{3} \\ -\frac{1}{3}y + \frac{2}{3} \end{pmatrix} \qquad P_7 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}x + 1 \\ \frac{1}{3}y + \frac{1}{3} \end{pmatrix}$$

$$P_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + 0 \\ \frac{1}{3}y + 0 \end{pmatrix} \qquad P_5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{1}{3} \\ -\frac{1}{3}y + \frac{1}{3} \end{pmatrix} \qquad P_6 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{pmatrix}$$

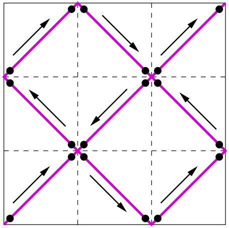
Approximating Polygons of the Peano Curve

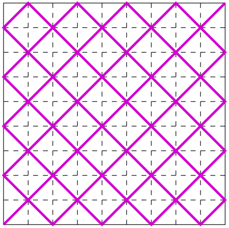
Definition:

The straight connection between the $9^n + 1$ points

$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \ldots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

is called ***n*-th approximating polygon of the Peano curve**





Peano's Representation of the Peano Curve

Definition: (Peano curve, original construction by G. Peano)

- each $t \in \mathcal{I} := [0,1]$ has a ternary representation

$$t = (0_3.t_1 t_2 t_3 t_4 \ldots)$$

- define the mapping $p \colon \mathcal{I} \to \mathcal{Q} := [0,1] \times [0,1]$ as

$$p(t) := \begin{pmatrix} 0_3.t_1\, k^{t_2}(t_3)\, k^{t_2+t_4}(t_5) \ldots \\ 0_3.k^{t_1}(t_2)\, k^{t_1+t_3}(t_4) \ldots \end{pmatrix}$$

  where $k(t_i) := 2 - t_i$ for $t_i = 0, 1, 2$ and k^j is the j-times concatenation of the function k

Peano's Representation of the Peano Curve (2)

Still to prove:

- p is independent of the ternary representation
- the Peano curve $p : \mathcal{I} \to \mathcal{Q}$ defines a space-filling curve.

Comments:

- the direction of "switchback" can be both vertical (see definition), horizontal, or mixed;
- actually, **272 different Peano curves** of the switchback type can be constructed using the same principles;
  For comparison: there are only two different 2D Hilbert curves
- in addition: 2 Peano-Meander curves

How Long are Approximating Polygons?

Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

$\Rightarrow$ length of the n-th polygon is $2^n \to \infty$ for $n \to \infty$.

Corollaries:

- the "length" of the Hilbert curve is not well defined
- instead, we can give an "area" of the Hilbert curve (1, the area of the unit square)

$\Rightarrow$ **Question: what's the dimension of a Hilbert curve?**

Fractal Dimension of Curves

Measuring the length of a curve:

- approx. the curve by a polygon with faces of length ϵ
  $\Rightarrow$ gives a measured length $L(\epsilon)$.
  (cmp. approximating polygons of a space-filling curve)
- in case of recursive repeat of a Leitmotiv:
  replace each units of length r by a polygon of length q, then
  $$L\left(\tfrac{\epsilon}{r}\right) = \frac{q}{r} L(\epsilon), \qquad L(1) := \lambda$$
- we obtain for the length $L(\epsilon)$:
  $$L(\epsilon) = \lambda \epsilon^{1-D}, \qquad \text{where} \quad D = \log_r q = \frac{\log q}{\log r}$$

Fractal Dimension of Curves (2)

Length of a recursively defined curve computed as

$$L(\epsilon) = \lambda\epsilon^{1-D}, \qquad \text{mit} \quad D = \log_r q = \frac{\log q}{\log r}$$

$\Rightarrow$ D is the **fractal dimension** of the curve

$\Rightarrow$ λ is the lenth w.r.t. that dimension

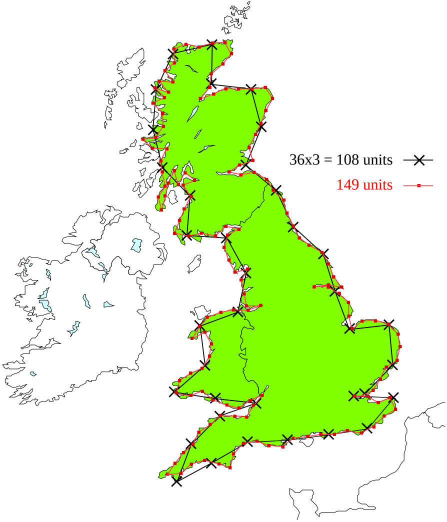
Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or ∞!
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

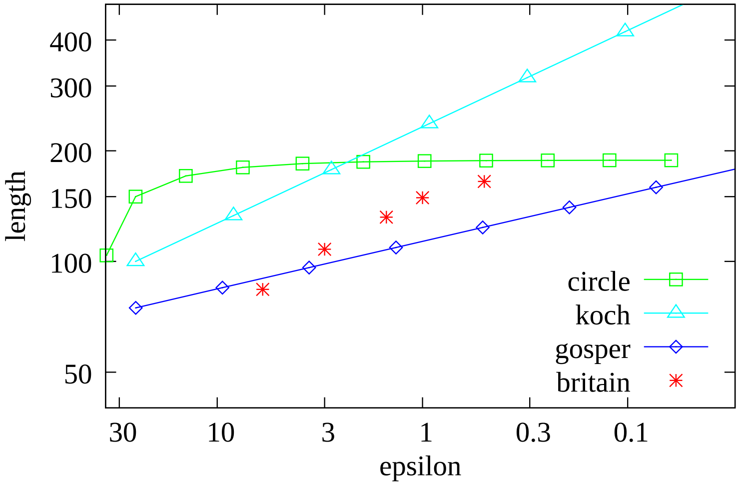
$\rightarrow$ **Hausdorff dimension**

How Long is the Coastline of Britain?

Compare, e.g., Mandelbrot: The Fractal Geometry of Nature

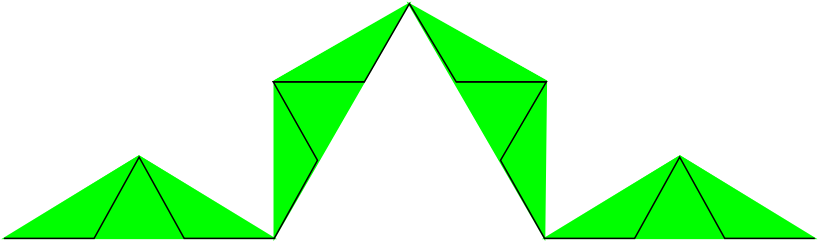


Test: Length of Fractal Curves



Exercise: What is the Area of a Fractal Curve?

Koch curve as example:



$\rightarrow$ refine green area and compute its limit value . . .