Algorithms of Scientific Computing

Space-Filling Curves in 3D

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Classification of Space-filling Curves

Definition: (*recursive* space-filling curve)

A space-filling curve $f\colon \mathcal{I} \to \mathcal{Q} \subset \mathbb{R}^n$ is called **recursive**, if both $\mathcal{I}$ and $\mathcal{Q}$ can be divided in m subintervals and sudomains, such that

- $f_*(\mathcal{I}^{(\mu)}) = \mathcal{Q}^{(\mu)}$ for all $\mu = 1, \ldots, m$, and
- all $\mathcal{Q}^{(\mu)}$ are geometrically similar to $\mathcal{Q}$.

Definition: (*connected* space-filling curve)

A recursive space-filling curve is called **connected**, if for any two neighbouring intervals $\mathcal{I}^{(\nu)}$ and $\mathcal{I}^{(\mu)}$ also the corresponding subdomains $\mathcal{Q}^{(\nu)}$ and $\mathcal{Q}^{(\mu)}$ are direct neighbours, i.e. share an $(n-1)$-dimensional hyperplane.

Connected, Recursive Space-filling Curves

Examples:

- all Hilbert curves (2D, 3D, ...)
- all Peano curves

Properties: connected, recursive SFC are

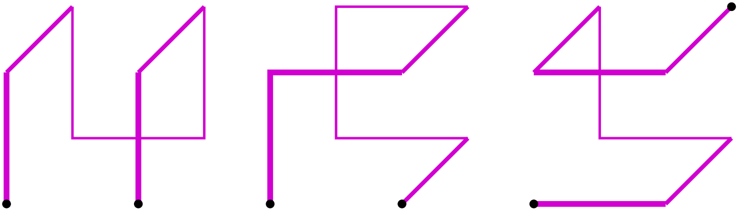
- continuous (more exact: Hölder continuous with exponent $1/n$)
- neighbourship-preserving
- describable by a grammar
- describable in an arithmetic form
  (similar to that of the Hilbert curve)

Related terms:

- face-connected, edge-connected, node-connected, ...
- also used for the induced orders on grid cells, etc.

3D Hilbert Curves

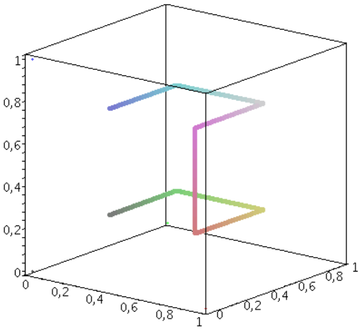
- Wanted: connected, recursive SFC, based on division-by-2
  $\Rightarrow$ leads to 3 basic patterns:



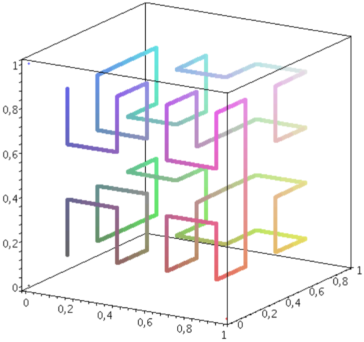
- in addition: symmetric forms, change of orientation
- always two different orientations of the components

$\Rightarrow$ numerous different Hilbert curves

3D Hilbert Curves – Iterations



1st iteration



2nd iteration

3D Hilbert Curve – Arithmetic Representation

t given in the octal system, $t = 0_8.k_1k_2k_3k_4\ldots$, then

$$h(0_8.k_1k_2k_3k_4\ldots) = H_{k_1} \circ H_{k_2} \circ H_{k_3} \circ H_{k_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with operators

$$H_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ \frac{1}{2}z + 0 \\ \frac{1}{2}y + 0 \end{pmatrix} \qquad H_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + 0 \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}x + 0 \end{pmatrix}$$

$$H_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + 0 \end{pmatrix} \qquad H_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}x + \frac{1}{2} \\ -\frac{1}{2}y + \frac{1}{2} \end{pmatrix}$$

3D Hilbert Curve – Arithmetic Representation (cont.)

$$H_4\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + 1 \\ -\frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix} \quad H_5\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + \frac{1}{2} \end{pmatrix}$$

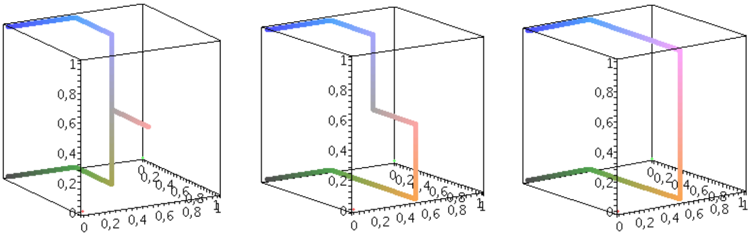
$$H_6\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ -\frac{1}{2}x + 1 \end{pmatrix} \quad H_7\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ -\frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}y + 1 \end{pmatrix}$$

$\Rightarrow$ leads to algorithm analog to $2D$ Hilbert and $2D$ Peano

$\Rightarrow$ uses only one pattern; each in only one orientation

3D Hilbert Curves – Variants

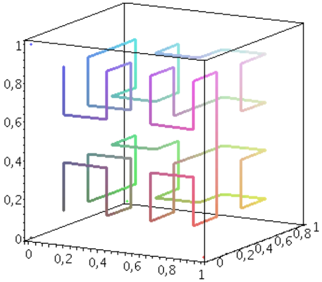
Different approximating polygons:

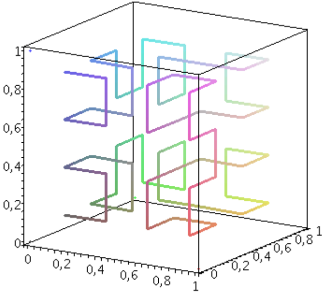


- same basic pattern:
  same order of the eight sub-cubes
- differences only noticeable from the 2nd iteration

3D Hilbert Curves – Variants (2)

Different orientation of the sub-cubes:

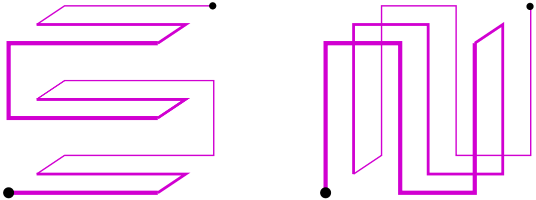




- same basic pattern Grundmotiv, same approximating polygon
- differences only visible from 2nd iteration

3D Peano Curves

- Concentration on “serpentine” Peano curves (no Meander-type)
- still lots of different variants
- especially interesting are dimension-recursive variants:



in each 3D cut, the sub-cubes are again traversed in Peano order

Parallelisation using Space-filling Curves

Problem setting:

- “mesh” (2D, 3D, ...) of N unknowns ($N \gg 1000$)
- solve linear system(s) of equations (maybe repeatedly with varying right-hand side)
- in the system, only spatially neighbouring unknowns are coupled

Parallelisation:

Distribute N unknowns to p partitions, such that

- each partition contains the same number of unknowns (*load balancing*)
- for as many unknowns as possible, all neighbours are in the same partition ($\Rightarrow$ avoids communication between partitions)

Parallelisation using Space-filling Curves (2)

Further demand: adaptivity

- add further unknowns (during/depending on intermediate results) or drop unknowns
- (re-)partitioning required to be **fast**:
  must not cost more computation time than going on with a bad load balance
- "shape preserving": if only few unknowns are added or dropped, the shape of partitions should not change strongly
  $\Rightarrow$ only few unknowns then need to migrate to another partition

$\Rightarrow$ popular strategy: use **space-filling curves**

Hölder Continuity of Space-filling Curves

Definition: (*Hölder continuous*)

A function f is called **Hölder continuous with exponent r** on the interval I, if a constant $C > 0$ exists, such that for all $x, y \in I$:

$$\|f(x) - f(y)\|_2 \leq C\,|x - y|^r$$

Importance for space-filling curves:

- $|x - y|$ is the distance of the indices
- $\|f(x) - f(y)\|$ is the distance of the image points (in "space")
- To prove: the Hilbert curve is Hölder coninuous with exponent $r = d^{-1}$, where d is the dimension:

$$\|f(x) - f(y)\|_2 \leq C\,|x - y|^{1/d} = C\sqrt[d]{|x - y|}$$

Hölder Continuity of the 3D Hilbert Curve

Proof analogous to simple continuity proof:

- given $x, y \in \mathcal{I}$; find an n, such that $8^{-(n+1)} < |x - y| < 8^{-n}$
- 8^{-n} is the interval lenth for the n-th iteration
  $\Rightarrow [x, y]$ covers at most two neighbouring(!) intervals.
- per construction of the 3D Hilbert curve, the function values $h(x)$ and $h(y)$ are in two adjacent cubes of side length 2^{-n}.
- the length of the space diagonal through the two adjacent cubes is $2^{-n} \cdot \sqrt{1^2 + 1^2 + 2^2} = 2^{-n} \cdot \sqrt{6}$, hence:

$$\begin{aligned}
\|h(x) - h(y)\|_2 &\leq 2^{-n}\sqrt{6} = \left(8^{-n}\right)^{1/3}\sqrt{6} = \left(8^{-(n+1)}\right)^{1/3} 8^{1/3}\sqrt{6} \\
&\leq 2\sqrt{6}\,|x - y|^{1/3} \qquad \text{q.e.d.}
\end{aligned}$$

Hölder Continuity and Parallelisation

- for the Hilbert curve (also Peano curve and all connected, recursive SFC), we have:

$$\|f(x) - f(y)\|_2 \leq C \sqrt[d]{|x - y|}$$

- relates the distance $|x - y|$ between indices to the disctance $\|f(x) - f(y)\|$ of (mesh) points
- gives relation between volume (number of grid cells/points) and extent (e.g. radius) of a partition

$\Rightarrow$ Hölder continuity gives a quantitative estimate for **compactness** of partitions