

# Algorithms of Scientific Computing

## 1 Project: Interpolation of the Trajectory of the Asteroid Pallas – Sample Solution

### Python Demo

see Python-Worksheet pallas1.py.

### Exercise 1

$$\begin{aligned} X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left( \cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \\ &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \left( \cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) + i \Im\{c_k\} \cdot \left( \cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \end{aligned}$$

Now we sort for real and imaginary part of  $X_l$ :

$$\begin{aligned} \Re\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\ \Im\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Im\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \end{aligned}$$

Since  $X_l$  is real  $\Im\{X_l\}$  must be zero. Thus, the remaining part is

$$X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

By reducing the sum to the range  $k = 1, \dots, \frac{N}{2} - 1$ , we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) \\ - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \Im\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} (\Re\{c_k\} + \Re\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) \\ + (\Im\{c_{-k}\} - \Im\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

Since  $c_{-k} = c_k^*$  it is  $\Re\{c_{-k}\} = \Re\{c_k\}$  and  $\Im\{c_{-k}\} = -\Im\{c_k\}$ . So, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

For  $N = 12$ ,  $a_k = 2\Re\{c_k\}$ ,  $b_k = -2\Im\{c_k\}$  for all  $k = 1, \dots, \frac{N}{2}$ ,  $a_0 = c_0$  and  $a_{\frac{N}{2}} = c_{\frac{N}{2}}$  we get equation (2):

$$X_l = a_0 + \sum_{k=1}^5 \left( a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right) \right) + a_6 \cos(\pi l)$$

## Python Demo

see Python-Worksheet pallas2.py.