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Worksheet 4
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Algorithms of Scientific Computing

Discrete Sine Transformation – Solution

Exercise 1: Discrete Sine Transform

a) Show that the corresponding Fourier coefficients are real

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \tag{1}$$

From the given $2N$ input data symmetry condition $f_{-n} = -f_n$ we find that $f_0 = -f_0$. This is possible only if $f_0 = 0$. For $n = N$ we get $f_{-N} = -f_N$, but due to the $2N$ periodicity requirement to the data $f_{n+2N} = f_n$ follows $f_{-N} = f_{-N+2N} = f_N$. Again this is possible only if $f_N = f_{-N} = 0$.

The proof is done in the following steps:

① Isolate the symmetry condition

② Insert the symmetry condition

③ Assemble terms to a sum over f_n

④ Make terms "imaginary"

$$
\begin{aligned}
F_k &= \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \\
&\overset{①}{=} \frac{1}{2N} \left(\sum_{n=-N+1}^{-1} f_n \omega_{2N}^{-kn} + f_0 \omega_{2N}^{0} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N \omega_{2N}^{-kN} \right) \\
&= \frac{1}{2N} \left(\sum_{n=1}^{N-1} f_{-n} \omega_{2N}^{kn} + f_0 e^{0} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N e^{-i2\pi kN/2N} \right) \\
&\overset{②}{=} \frac{1}{2N} \left(- \sum_{n=1}^{N-1} f_n \omega_{2N}^{kn} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} \right) \\
&\overset{③}{=} \frac{1}{2N} \sum_{n=1}^{N-1} f_n \underbrace{\left(\omega_{2N}^{-kn} - \omega_{2N}^{kn} \right)}_{=\omega_{2N}^{-kn} - \left(\omega_{2N}^{kn}\right)^{*} = -2i\mathrm{Im}\{\omega_{2N}^{kn}\}} \\
&\overset{④}{=} \frac{-2i}{2N} \sum_{n=1}^{N-1} f_n \underbrace{\mathrm{Im}\{e^{i2\pi kn/2N}\}}_{=\mathrm{Im}\{\cos\left(\frac{\pi kn}{N}\right) + i\sin\left(\frac{\pi kn}{N}\right)\}} \\
&= \frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi kn}{N}\right) \in \mathbb{R} \qquad \text{q.e.d.}
\end{aligned}
$$

b) Show that the F_k also have a symmetry:

Since there are only sine terms we assume (and get) an odd symmetry:

$$
\begin{aligned}
F_{-k} &= \frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{-\pi kn}{N}\right) \\
&= -\frac{-i}{N} \sum_{n=1}^{N-1} f_n \sin\left(\frac{\pi kn}{N}\right) \\
&= -F_k
\end{aligned}
$$

Since all $F_{-k} = -F_k$, we need the F_k only for $k = 0, \ldots, N$ for a Sine Transform.

c) Algorithm for the Sine Transform

The procedure `FFT(f,N)` computes the correct coefficients, if we pass the $N+1$ data from field `g` as a dataset of length $2N$ with symmetry $f_{-n} = -f_n$.

From equation (1) of the worksheet we know that `FFT(f,N)` gets a dataset `f` with indices $n = -N+1, \ldots, N$. We only have to compute the F_k for $k = 1, \ldots, N-1$.

So, the algorithm looks like this:

1. Set `f[0] := 0` $= f_0$
   For all $n = 1, \ldots, N-1$:
      Set `f[n] := g[n]` $= f_n$
      Set `f[-n] := -g[n]` $= f_n$
   Set `f[N] := 0` $= f_N$
2. Call `FFT(f,`N`)`
3. *(Now the Fourier coefficients F_k are stored in the field `f`)*
   For all $k = 1, \ldots, N-1$:
      Set `g[k] := f[k]` $= F_k$

Exercise 2: Fast Discrete Sine Transform

The butterfly scheme is retrieved as usual:

$$
\begin{aligned}
F_k &= \frac{1}{2N}\sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} = \frac{1}{2}\left(\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n}\omega_{2N}^{-2kn} + \frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n-1}\omega_{2N}^{-k(2n-1)}\right) \\
&= \frac{1}{2}\Bigg(\underbrace{\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n}\omega_N^{-kn}}_{=:G_k} + \underbrace{\frac{1}{N}\sum_{n=-N/2+1}^{N/2} f_{2n-1}\omega_N^{-kn}}_{=:H_k}\,\omega_{2N}^{k}\Bigg) \\
&= \frac{1}{2}\left(G_k + \omega_{2N}^k H_k\right) \\
F_{k+N} &= \frac{1}{2}\left(G_{k+N} + \omega_{2N}^{k+N} H_{k+N}\right) = \frac{1}{2}\left(G_k - \omega_{2N}^k H_k\right)
\end{aligned}
$$

For the datasets $g_n := f_{2n}$ and $h_n := f_{2n-1}$, respectively, we can try to find other symmetries:

$$g_{-n} = f_{2(-n)} = -f_{-2n} = -f_{2n} = -g_n$$

The "odd" data also shows an odd symmetry and therefore lead to another Sine Transform but with half length.

Analog for the data with odd indices:

$$h_{-n} = f_{2(-n)-1} = f_{-2n-1} = -f_{2n+1} = -f_{2(n+1)-1} = -h_{n+1}$$

Again we get an "odd" symmetry. However, this is the transform shown in the lecture, known as Quarter-Wave-DST, again with half length.

For a dataset with the symmetry constraint $f_{-n} = -f_{n+1}$ we get accordingly

$$g_{-n} = f_{2(-n)} = f_{-2n} = -f_{2n+1} = -h_{n+1}$$

and

$$h_{-n} = f_{-2n-1} = f_{-2n+1} = -f_{2n+2} = -f_{2n-1} = -g_{n+1}$$

Exercise 3: DFT and Least Square Approximation

We set all partial of E derivatives to zero:

$$\sum_{n=-(N-1)/2}^{(N-1)/2} \left[e^{-i2\pi nk/N} \left(f_n - \sum_{p=-(N-1)/2}^{(N-1)/2} \alpha_p e^{i2\pi np/N} \right) \right] = 0. \tag{2}$$

Rearranging the terms gives us the set of N equations

$$\sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk} = \sum_{p=-(N-1)/2}^{(N-1)/2} \alpha_p \sum_{n=-(N-1)/2}^{(N-1)/2} \omega_N^{n(p-k)}, \tag{3}$$

where $\omega_N = e^{i2\pi/N}$.

Next, we will find the second sum in the right hand side of these equations. We notice that N complex numbers ω_N^k, for $k = 0 : N-1$ are the Nth roots of unity because they satisfy

$$(\omega_N^k)^N = \left(e^{i2\pi k/N}\right)^N = e^{i2\pi k} = 1 \tag{4}$$

and therefore are zeros of the polynomial $z^N - 1$. We can factor this polynomial as

$$z^N - 1 = (z-1)\left(z^{N-1} + z^{N-2} + \ldots + z + 1\right) = (z-1)\sum_{n=0}^{N-1} z^n. \tag{5}$$

If $z = \omega_N^{j-k}$, where $j-k$ is not multiple of N, then $z \neq 1$, and we thus have

$$\sum_{n=0}^{N-1} z^n = \sum_{n=0}^{N-1} \omega_N^{(j-k)n} = 0. \tag{6}$$

On the other hand, if $j-k$ is multiple of N then $\omega_N^{j-k} = 1$ and

$$\sum_{n=0}^{N-1} z^n = \sum_{n=0}^{N-1} \omega_N^{(j-k)n} = \sum_{n=0}^{N-1} 1 = N. \tag{7}$$

Since the sequence ω_N^k is N-periodic we get

$$\sum_{n=-(N-1)/2}^{(N-1)/2} \omega_N^{n(p-k)} = N\delta_N(p-k), \tag{8}$$

where $\delta_N(k)$ is known as the modular Kronecker delta. It is defined by

$$\delta_N(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or a multiple of } N, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Using this result in equation (3) we obtain

$$\sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk} = N\alpha_k, \tag{10}$$

for $k = -(N-1)/2 : (N-1)/2$. We get that α_ks correspond exactly to the DFT coefficients

$$\alpha_k = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk}. \tag{11}$$