Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
M.Sc. K. Röhner, M.Sc. D. Jarema

SS 2014
Sheet 8
2.6.2014

Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
Hierarchization in Higher Dimensions, Combination Technique

Tutorial 8: Monday, June 2nd 2014, Room MI 02.07.023 *Proposed solution*

Exercise 1: Hierarchization in Higher Dimensions

In this exercise we will implement the multi-recursive algorithm for hierarchization of a multi-dimensional regular sparse grid. The structure of the code resembles strongly the one-dimensional case, and so we again have a class (`PagodaFunction`) representing our grid points.

(i) Implement the refinement criterion `MinLevelCriterion` that adds all points up to a specified level to a given grid.
Hint: In your grid traversal, try to avoid multiple visits to the same grid points.

(ii) Implement the function `hierarchize` efficiently using a recursive approach.
Hint: The underlying traversal algorithm can be implemented similar to the one in (i).

(iii) Implement a function to compute the volume of the sparse grid interpolant.

For the solution code look at the pdf exported from IPython Notebook.

Exercise 2: The Combination Technique – A Different View on Sparse Grids

Dealing with hierarchical bases often turns out to be sophisticated. On this worksheet we will therefore see how the so-called *combination technique* finds a sparse grid interpolant, that approximates a function on a number of full grids, each consisting only of a "relatively small" number of grid points.

Let $u_{\underline{l}}$ ($\underline{l} \in \mathbb{N}^2$) for a $u : [0,1]^2 \to \mathbb{R}$ the interpolant in $V_{\underline{l}}$ (interpolating piecewise bilinearly at the inner grid points, at the boundary u is assumed to be zero again).

(i) $V_{\underline{l}}$ can be decomposed into a set of subspaces $W_{\underline{l}}$. Accordingly, the interpolant $u_{\underline{l}} \in V_{\underline{l}}$ can be written as a sum of $w_{\underline{l}} \in W_{\underline{l}}$.

Spot the grid associated with $u_{(3,2)}$ in the right part of Figure 1. Identify those subspaces in the left part that are needed to reconstruct $u_{(3,2)}$.

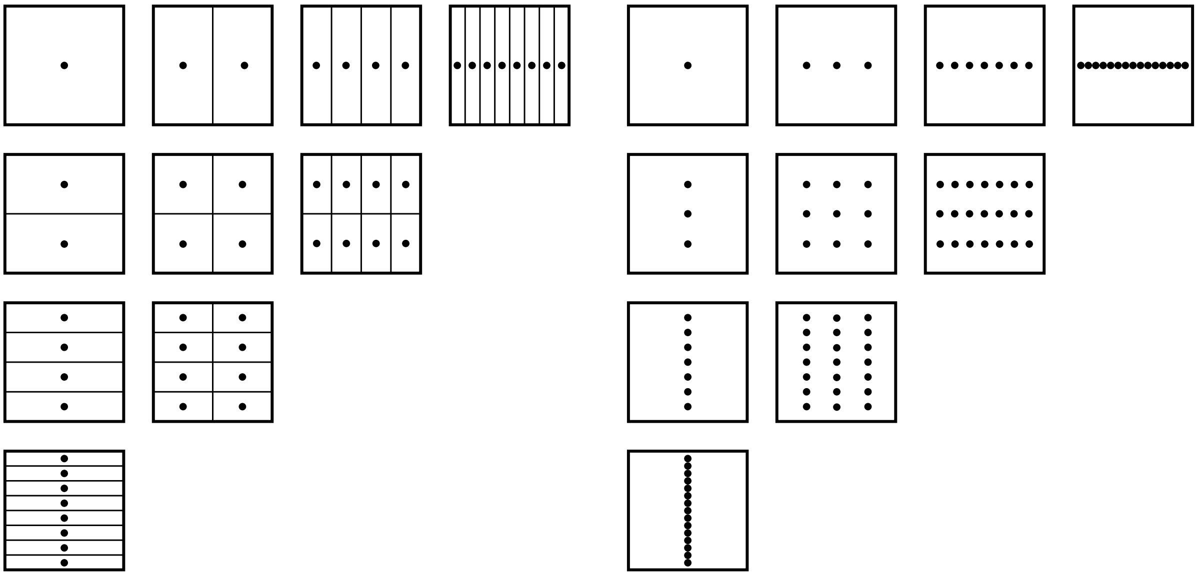

Figure 1: The two parts in the picture show the grid points and supports associated with interpolants $w_{\underline{l}}$ (left) and $u_{\underline{l}}$ (right) up to level 4 for the 2d case.

We need to "collect" those $w_{\underline{l}}$ with indices bound component-wise by $\underline{l}$. In subspace scheme (see left hand side of Figure 1) these are the subspaces in the rectangular left upper part relative to $\underline{l}$. Written as a sum this gives us:

$$u_{\underline{l}} = \sum_{\underline{l}' \leq \underline{l}} w_{\underline{l}'}$$

Note that weights are not needed yet.

(ii) Use the result from (i) to rewrite

$$\sum_{|\underline{l}|_1 = n+1} u_{\underline{l}} \,, \qquad n \in \mathbb{N}$$

for the two-dimensional case as a weighted sum of $w_{\underline{l}}$.

Hint: Look at the subspace scheme in Figure 1 and count the occurrences of each subspace in the sum. What do you notice when comparing $w_{\underline{l}}$ with common level $n = |\underline{l}|_1 + dim - 1$?

Using the previous result we start with

$$\sum_{|\underline{l}|_1 = n+1} u_{\underline{l}} = \sum_{|\underline{l}|_1 = n+1} \sum_{\underline{l}' \leq \underline{l}} w_{\underline{l}'}.$$

For the reorganization part we first count which $w_{\underline{l}'}$ appears how many times in the sums. (remember, this is **2d***!)*

$|\underline{l}'|_1 = n+1$*:* *each $w_{\underline{l}}$ has one occurrence*

$|\underline{l}'|_1 = n$*:* *each $w_{\underline{l}}$ has two occurrences*

$\vdots$

$|\underline{l}'|_1 = k$: *each has $n+2-k$ occurrences*

Technical explanation: Let $\underline{l}$ be of level $n+1$, i.e. $|\underline{l}|_1 = n+1$. From $\underline{l}$ construct all possible $\underline{l}'$ of level n with $\underline{l}' \leq \underline{l}$. Those are exactly two, as there are two components (2d!) that can be decreased by 1. Applying this scheme down to level 1 then leads to the result above.

Written as an explicit sum formula this is:

$$\sum_{|\underline{l}|_1=n+1} u_{\underline{l}} = \sum_{|\underline{l}|_1\leq n+1} (n+2-|\underline{l}|_1) w_{\underline{l}}.$$

(iii) In the final step use the previous results to give a representation of the sparse grid interpolant

$$u_n^D := \sum_{|\underline{l}|_1\leq n+1} w_{\underline{l}}$$

as a weighted sum of $u_{\underline{l}}$. Again, count the occurrences of the $w_{\underline{l}}$.

By looking at the subspace scheme rather than by looking at the formulas it becomes clear that the following holds:

$$\sum_{|\underline{l}|_1\leq n+1} w_{\underline{l}} = \sum_{|\underline{l}|_1=n+1} u_{\underline{l}} - \sum_{|\underline{l}|_1=n} u_{\underline{l}}$$

(iv) Assume you are talking to a person who knows how to approximate the volume $F_2(u)$ through the trapezoidal rule (in 2d) with respect to $u_{\underline{l}}$. Give instructions on how to write a program that implements a sparse grid approximation of $F_2(u)$. Remember Archimedes quadrature.

- *First idea: Replace volume $F_2(u)$ by the sparse grid volume approximation $F_2(u_n^D)$.*
- *Second idea: Think of the interpolant as a sum of $u_{\underline{l}}$. We know those $u_{\underline{l}}$ (interpolating u on regular grids) as well as their volumes (trapezoidal rule in 2d).*
- *Together with the weights from the previous part we get*

$$F_2(u) \approx F_2(u_n^D) = \sum_{|\underline{l}|_1=n+1} F_2(u_{\underline{l}}) - \sum_{|\underline{l}|_1=n} F_2(u_{\underline{l}})$$

(v) Compare this method with Archimedes quadrature — what are the (dis-)advantages?

Advantages:

- *Simpler program code (Haven't you tried coding them? Do it! You'll agree...)*
- *It might be possible to reuse an existing program for the trapezoidal rule on common regular grids (advantage is even bigger for more complex applications, e.g. when computing a sparse grid solution for a fluid simulation)*
- *For comprehensive computations the program is more likely and easy to be parallelized as the single grids are processed independently from each other*

Disadvantages:

- *No straight forward approach to include adaptivity, i.e. it's not possible to automatically find the right evaluation points*
- *Recursion is much more beautiful!*