

Algorithms of Scientific Computing

Hierarchical Methods and Sparse Grids

– 1D Hierarchical Basis –

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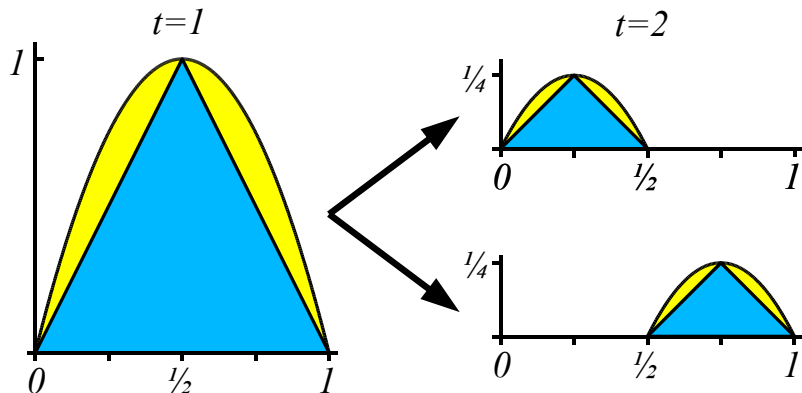
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Summer Term 2014



Archimedes' Quadrature

Compute an approximation of $F_1 := \int_0^1 4 \cdot x \cdot (1 - x) dx = \frac{2}{3}$



Archimedes' Quadrature (2)

- Integrating $4x(1 - x)$, we have to consider several quantities
- Ordered by (recursive) level t :

Level-depth	1	2	3	4	...	t
Mesh-width h	$1/2$	$1/4$	$1/8$	$1/16$...	2^{-t}
# triangles	1	2	4	8	...	$\frac{1}{2}2^t$
surplus v	1	$1/4$	$1/16$	$1/64$...	$4 \cdot 2^{-2t}$
Area of triangle D_1	$1/2$	$1/16$	$1/128$	$1/1024$...	$4 \cdot 2^{-3t}$
Sum (current t)	$1/2$	$1/8$	$1/32$	$1/128$...	$2 \cdot 2^{-2t}$
Sum ($\leq t$)	$1/2$	$5/8$	$21/32$	$85/128$...	$\frac{2}{3} (1 - 2^{-2t})$
Error	$1/6$	$1/24$	$1/96$	$1/384$...	$\frac{2}{3}2^{-2t}$

Approximation of Functions

- To analyze Archimedes' quadrature rule, we consider functions
- We need a representation of the (approximating) function $u(x)$ which we are integrating:
 - u as linear combination of ansatz functions ϕ_j :

$$u(x) = \sum_{i=1}^n \alpha_i \cdot \phi_i(x)$$

- Integrating $u(x)$:

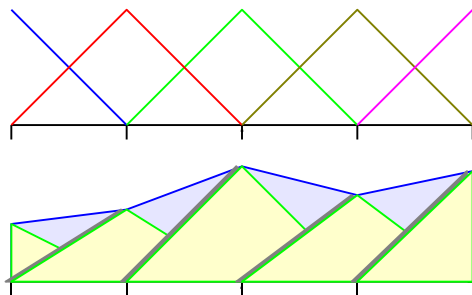
$$\int_a^b u(x) dx = \sum_i^n \alpha_i \int_a^b \phi_i(x) dx,$$

- Weighted sum of α_j
- Remember: Newton-Cotes formulas are weighted sum of function evaluations

Composite Trapezoidal Rule: Function

Interpolant

- Continuous, piecewise linear function
- Represent u in nodal point (hat) basis



- Coefficients α_j are function values at grid points
- Basis functions have area h ($h/2$ at boundaries)

Piecewise Linear Functions

Ansatz space and basis functions

- Only consider $u : [0, 1] \rightarrow \mathbb{R}$
- Consider discretization level $n \in \mathbb{N}$
- Mesh-width $h_n = 2^{-n}$
- Grid points $x_{n,i} = i \cdot h_n$
- Define “mother of all hat functions”

$$\phi(x) := \max\{1 - |x|, 0\}$$

⇒ Basis functions

$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- Nodal point basis $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$

Piecewise Linear Functions (2)

- Space of continuous piecewise linear functions

$$V_n = \text{span}(\Phi_n)$$

- Interpolants $u_n \in V_n$

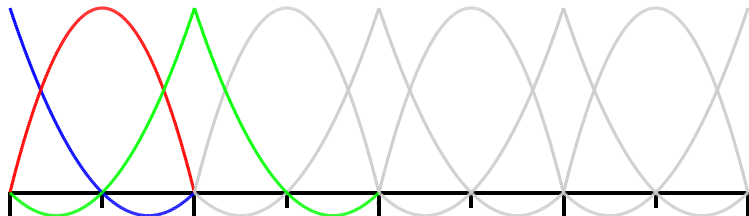
$$u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i} \phi_{n,i}(x)$$

- V_n the space of all such interpolants u_n

Composite Simpson's Rule: Function

Interpolant

- Continuous, piecewise quadratic function
- More complicated basis:



- Basis functions: Lagrangian polynomials, glued together
- α_j : function values at grid points
- Basis functions have area $h/6$ (blue), $4h/6$ (red), $2h/6$ (green)
- We'll not formally define basis functions here. . .

From Composite Trapezoidal to Archimedes

Piecewise linear functions

- We restrict our functions u to $u(0) = u(1) = 0$
- Nodal point basis for discretization level n :

$$\Phi_n := \{\phi_{n,i}, 1 \leq i \leq 2^n - 1\}$$

- Wanted: *function space*

$$V := \bigcup_{l=1}^{\infty} V_l$$

contains all functions which are in V_l for sufficiently large l

- However: generating system of V as

$$\Phi := \bigcup_{l=1}^{\infty} \Phi_l$$

does not lead to a basis (not linear independent)

Hierarchical Basis

- We are interested in a hierarchical decomposition of V_l
- Define **hierarchical increment** W_l , such that V_l is a *direct sum*:

$$V_l = V_{l-1} \oplus W_l$$

Side-note: direct sum

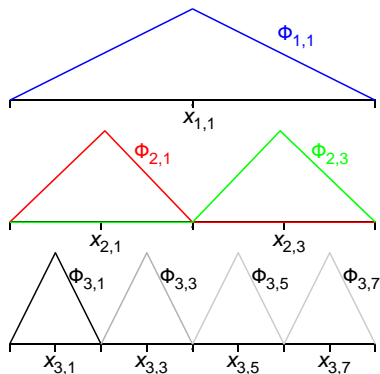
- Every $u_l \in V_l$ can be uniquely decomposed as $u_l = u_{l-1} + w_l$, with $u_{l-1} \in V_{l-1}$ and $w_l \in W_l$
- W_l has to contain 2^{l-1} ansatz functions:
 $\dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l$
- This holds (introducing index sets \mathcal{I}_l) for

$$\mathcal{I}_l := \{i : 1 \leq i < 2^l, i \text{ odd}\}$$

$$W_l := \text{span} \{\phi_{l,i} : i \in \mathcal{I}_l\}$$

Hierarchical Increments

- Set of hierarchical increments W_l
- For $l = 1$: $W_1 = V_1$
- Example for $l = 1, 2, 3$:



Hierarchical Basis (cont.)

- Then

$$V_n = \bigoplus_{l=1}^n W_l$$

is a direct sum, too:

- $u \in V_n$ can be decomposed uniquely into $w_l \in W_l$:

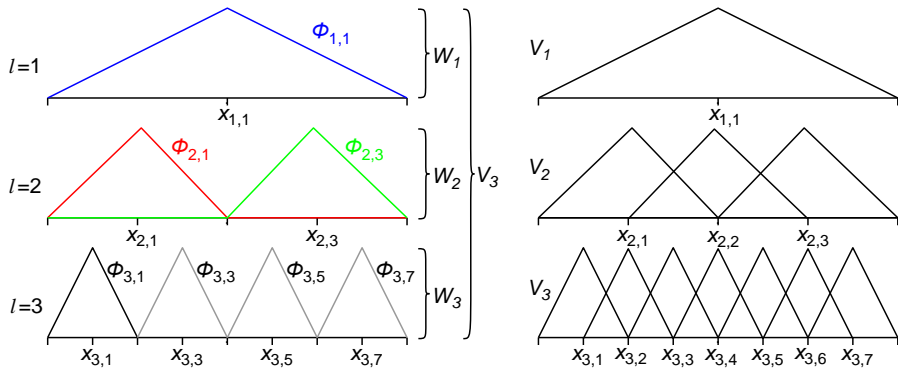
$$u = \sum_{l=1}^n w_l = \sum_{l=1}^n \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i}$$

→ Coefficients $v_{l,i}$ are hierarchical surplusses

- Corresponding basis of V_n (or, with ∞ instead of n , of V)

$$\Psi_n := \bigcup_{l=1}^n \{\phi_{l,i} : i \in \mathcal{I}_l\}.$$

Comparison



Comparison (2)

