Algorithms of Scientific Computing

Space-Filling Curves – Approximating Polygons and Fractals

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Approximating Polygons of the Hilbert Curve

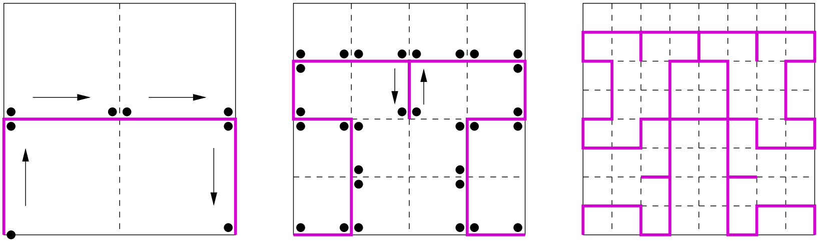
Idea: Connect start and end point of iterate on each subcell.

Definition:

The straight connection of the $4^n + 1$ points

$$h(0), h(1 \cdot 4^{-n}), h(2 \cdot 4^{-n}), \ldots, h((4^n - 1) \cdot 4^{-n}), h(1)$$

is called the ***n*-th approximating polygon of the Hilbert curve**



Properties of the Approximating Polygon

- the approximating Polygon connects the start and end points of the space-filling curve within each subsquare

  ⇒ **assists in the construction of space-filling curves**

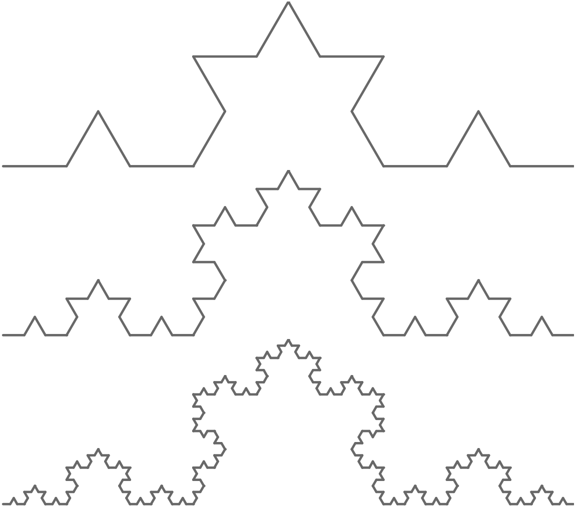
- usually the approximating Polygon connects **corners**
- approximating polygons are constructed by recursive repetition of a so-called **Leitmotiv**

  ⇒ **similarity to Koch and other fractal curves**

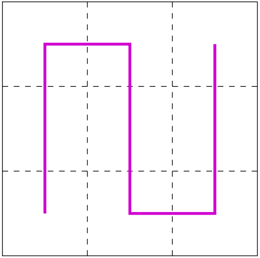
- the sequence of corresponding functions $p_n(t)$ converges **uniformly** towards h

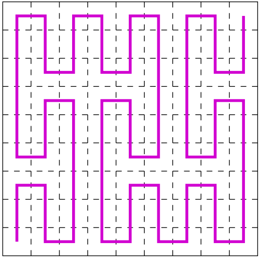
  ⇒ additional proof of continuity of the Hilbert curve

Example: Koch Curve



Construction of the Peano Curve





Recursive Construction:

- divide quadratic domain into 9 subsquares
- construct Peano curve for each subsquare
- join the partial curves to build a higher level curve

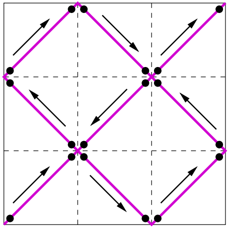
Approximating Polygons of the Peano Curve

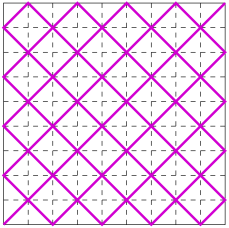
Definition:

The straight connection between the $9^n + 1$ points

$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \ldots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

is called **n-th approximating polygon of the Peano curve**





How Long are Approximating Polygons?

Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

  $\Rightarrow$ length of the n-th polygon is $2^n \to \infty$ for $n \to \infty$.

Corollaries:

- the "length" of the Hilbert curve is not well defined
- instead, we can give an "area" of the Hilbert curve (1, the area of the unit square)

$\Rightarrow$ **Question: what's the dimension of a Hilbert curve?**

Fractal Dimension of Curves

Measuring the length of a curve:

- approximate the curve by a polygon with faces of length ϵ
  $\Rightarrow$ gives a measured length $L(\epsilon)$.
- **cmp. approximating polygons of a space-filling curve**
  $\rightarrow$ measuring polygon through corners
- general case $\rightarrow$ recursive repetition of a Leitmotiv:
  replace each segment of length r [units] by a polygon of length q,

$$\text{then} \quad L\left(\tfrac{\epsilon}{r}\right) = \frac{q}{r} L(\epsilon), \qquad L(1) := \lambda$$

- we obtain for the length $L(\epsilon)$:

$$L(\epsilon) = \lambda \epsilon^{1-D}, \qquad \text{where} \quad D = \log_r q = \frac{\log q}{\log r}$$

Fractal Dimension of Curves (2)

Length of a recursively defined curve computed as

$$L(\epsilon) = \lambda \epsilon^{1-D}, \qquad \text{mit} \quad D = \log_r q = \frac{\log q}{\log r}$$

$\Rightarrow$ D is the **fractal dimension** of the curve

$\Rightarrow$ λ is the lenth w.r.t. that dimension

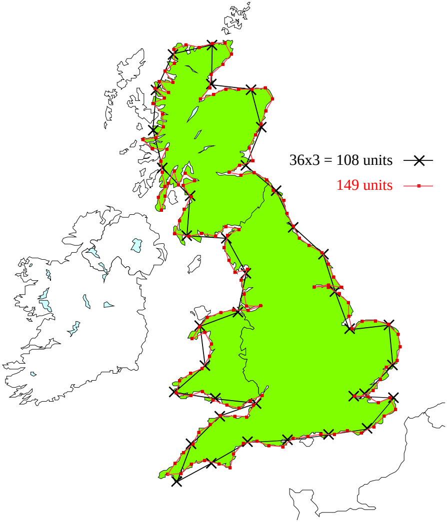
Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or ∞!
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

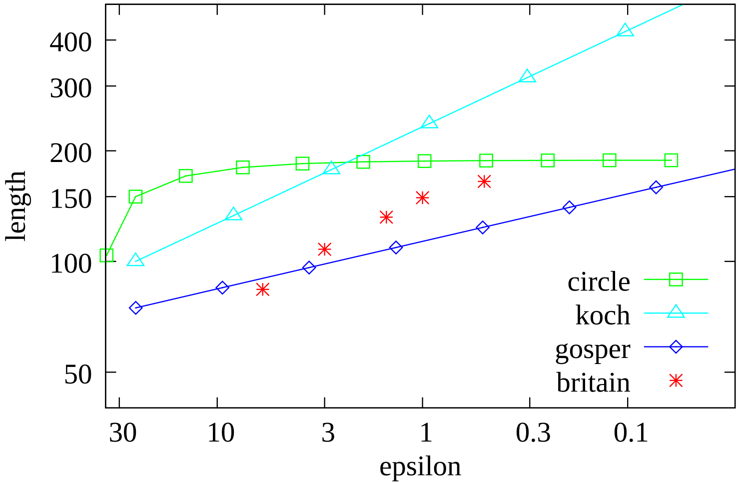
$\rightarrow$ **Hausdorff dimension**

How Long is the Coastline of Britain?

Compare, e.g., Mandelbrot: The Fractal Geometry of Nature

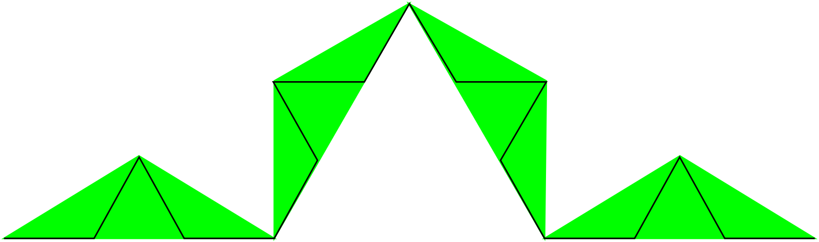


Test: Length of Fractal Curves



Exercise: What is the Area of a Fractal Curve?

Koch curve as example:



→ refine green area and compute its limit value . . .