Algorithms of Scientific Computing

Space-Filling Curves in 3D

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Classification of Space-filling Curves

Definition: (*recursive* space-filling curve)

A space-filling curve $f\colon \mathcal{I} \to \mathcal{Q} \subset \mathbb{R}^n$ is called **recursive**, if both $\mathcal{I}$ and $\mathcal{Q}$ can be divided in m subintervals and sudomains, such that

- $f_*(\mathcal{I}^{(\mu)}) = \mathcal{Q}^{(\mu)}$ for all $\mu = 1, \ldots, m$, and
- all $\mathcal{Q}^{(\mu)}$ are geometrically similar to $\mathcal{Q}$.

Definition: (*connected* space-filling curve)

A recursive space-filling curve is called **connected**, if for any two neighbouring intervals $\mathcal{I}^{(\nu)}$ and $\mathcal{I}^{(\mu)}$ also the corresponding subdomains $\mathcal{Q}^{(\nu)}$ and $\mathcal{Q}^{(\mu)}$ are direct neighbours, i.e. share an $(n-1)$-dimensional hyperplane.

Connected, Recursive Space-filling Curves

Examples:

- all Hilbert curves (2D, 3D, ...)
- all Peano curves

Properties: connected, recursive SFC are

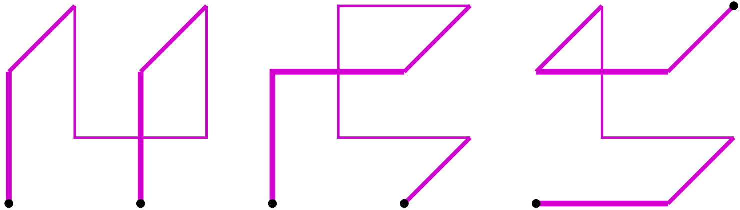
- continuous (more exact: Hölder continuous with exponent $1/n$)
- neighbourship-preserving
- describable by a grammar
- describable in an arithmetic form
  (similar to that of the Hilbert curve)

Related terms:

- face-connected, edge-connected, node-connected, ...
- also used for the induced orders on grid cells, etc.

3D Hilbert Curves

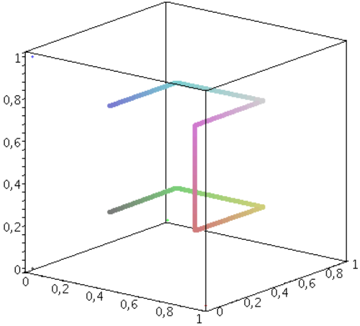
- Wanted: connected, recursive SFC, based on division-by-2
  $\Rightarrow$ leads to 3 basic patterns:



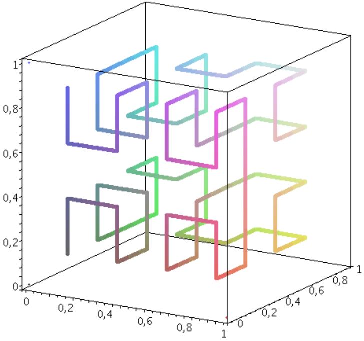
- in addition: symmetric forms, change of orientation
- always two different orientations of the components

$\Rightarrow$ numerous different Hilbert curves

3D Hilbert Curves – Iterations



1st iteration



2nd iteration

3D Hilbert Curve – Arithmetic Representation

t given in the octal system, $t = 0_8.k_1k_2k_3k_4\ldots$, then

$$h(0_8.k_1k_2k_3k_4\ldots) = H_{k_1} \circ H_{k_2} \circ H_{k_3} \circ H_{k_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with operators

$$H_0\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ \frac{1}{2}z + 0 \\ \frac{1}{2}y + 0 \end{pmatrix} \qquad H_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + 0 \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}x + 0 \end{pmatrix}$$

$$H_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + 0 \end{pmatrix} \qquad H_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}x + \frac{1}{2} \\ -\frac{1}{2}y + \frac{1}{2} \end{pmatrix}$$

3D Hilbert Curve – Arithmetic Representation
(continued)

$$H_4\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}z+1\\-\frac{1}{2}x+\frac{1}{2}\\\frac{1}{2}y+\frac{1}{2}\end{pmatrix}\quad H_5\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}x+\frac{1}{2}\\\frac{1}{2}y+\frac{1}{2}\\\frac{1}{2}z+\frac{1}{2}\end{pmatrix}$$

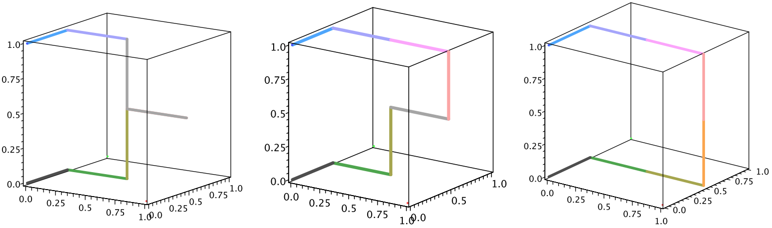
$$H_6\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}z+\frac{1}{2}\\\frac{1}{2}y+\frac{1}{2}\\-\frac{1}{2}x+1\end{pmatrix}\quad H_7\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}x+0\\-\frac{1}{2}z+\frac{1}{2}\\-\frac{1}{2}y+1\end{pmatrix}$$

$\Rightarrow$ leads to algorithm analog to $2D$ Hilbert and $2D$ Peano

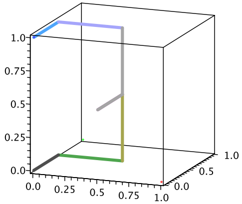
$\Rightarrow$ uses only one pattern; each in only one orientation

3D Hilbert Curves – Variants

Four different approximating polygons:

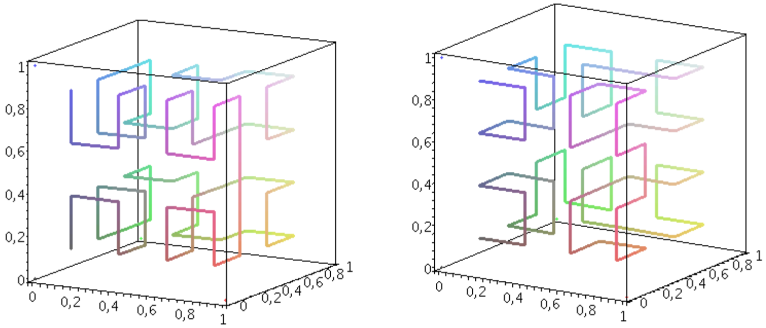


- same basic pattern:
  same order of the eight sub-cubes
- differences only noticeable
  from the 2nd iteration



3D Hilbert Curves – Variants (2)

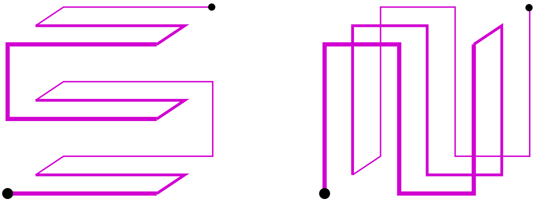
Two different orientations for each sub-cube:



- same basic pattern Grundmotiv, same approximating polygon
- differences only visible from 2nd iteration

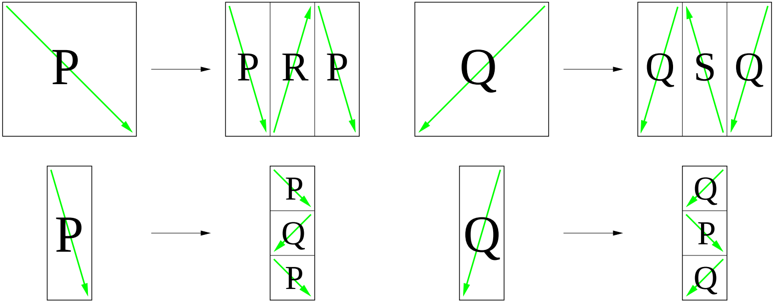
3D Peano Curves

- Concentration on “serpentine” Peano curves (no Meander-type)
- still lots of different variants
- especially interesting are dimension-recursive variants:



in each 3D cut, the sub-cubes are again traversed in Peano order

Peano Curve – Dimension-Recursive Grammar



Leads to Grammar: (analogous in 3D)

$$\begin{aligned}
P &\longleftarrow P_y \rightarrow R_y \rightarrow P_y & \qquad P_y &\longleftarrow P \uparrow Q \uparrow P \\
Q &\longleftarrow Q_y \leftarrow S_y \leftarrow Q_y & \qquad Q_y &\longleftarrow Q \uparrow P \uparrow Q \\
R &\longleftarrow R_y \rightarrow P_y \rightarrow R_y & \qquad R_y &\longleftarrow R \downarrow S \downarrow R \\
S &\longleftarrow S_y \leftarrow Q_y \leftarrow S_y & \qquad S_y &\longleftarrow S \downarrow R \downarrow S
\end{aligned}$$

Peano – Dimension-Recursive Arithmetization

Consider Parameter in Ternary Representation:

$$p(0_3.t_1 t_2 t_3 t_4 \ldots) = P^x_{t_1} \circ P^y_{t_2} \circ P^x_{t_3} \circ P^y_{t_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Rotation Operators: (here: only x-operators)

$$\begin{aligned}
P^x_0 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + 0 \\ \frac{1}{3}y + 0 \end{pmatrix} \\
P^x_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -x + 1 \\ \frac{1}{3}y + \frac{1}{3} \end{pmatrix} \\
P^x_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + 0 \\ \frac{1}{3}y + \frac{2}{3} \end{pmatrix}.
\end{aligned}$$

Peano's Representation of the Peano Curve

Definition: (Peano curve, original construction by G. Peano)

- each $t \in \mathcal{I} := [0,1]$ has a ternary representation

$$t = (0_3.t_1 t_2 t_3 t_4 \ldots)$$

- define the mapping $p\colon \mathcal{I} \to \mathcal{Q} := [0,1] \times [0,1]$ as

$$p(t) := \begin{pmatrix} 0_3.t_1\, k^{t_2}(t_3)\, k^{t_2+t_4}(t_5) \ldots \\ 0_3.k^{t_1}(t_2)\, k^{t_1+t_3}(t_4) \ldots \end{pmatrix}$$

  where $k(t_i) := 2 - t_i$ for $t_i = 0, 1, 2$ and k^j is the j-times concatenation of the function k

Comments:

- equivalent to dimension-recursive arithmetization
- thus: independent of the ternary representation

Variants of Space Filling Curves

Hilbert Curve:

- essentially only one 2D Hilbert curve (additional variants only due to rotations/symmetry or different start/end)
- **1024 different 3D Hilbert curves**
  (4 different approximating polygons, 2^8 different orientations)

Peano Curve:

- the direction of “switchback” can be both vertical (see definition), horizontal, or mixed
- **272 different 2D Peano curves** of the switchback type can be constructed using the same principles
- in addition: two 2D Peano-Meander curves
- huge number of variants in 3D

“Simple Curve”: requires only one set of operators