Algorithms of Scientific Computing

Parallelisation Using Space-Filling Curves

Michael Bader

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Parallelisation using Space-filling Curves

Problem setting:

- "mesh" (2D, 3D, ...) of N unknowns ($N \gg 1000$)
- solve linear system(s) of equations (maybe repeatedly with varying right-hand side)
- in the system, only spatially neighbouring unknowns are coupled

Parallelisation:

Distribute N unknowns to p partitions, such that

- each partition contains the same number of unknowns (*load balancing*)
- for as many unknowns as possible, all neighbours are in the same partition ($\Rightarrow$ avoids communication between partitions)

Parallelisation using Space-filling Curves (2)

Further demand: adaptivity

- add further unknowns (during/depending on intermediate results) or drop unknowns
- (re-)partitioning required to be **fast**:
  must not cost more computation time than going on with a bad load balance
- "shape preserving": if only few unknowns are added or dropped, the shape of partitions should not change strongly
  $\Rightarrow$ only few unknowns then need to migrate to another partition

$\Rightarrow$ popular strategy: use **space-filling curves**

Generic Space-filling Heuristic

Bartholdi & Platzman (1988):

1. Transform the problem in the unit square, via a space-filling curve, to a problem on the unit interval
2. Solve the (easier) problem on the unit interval

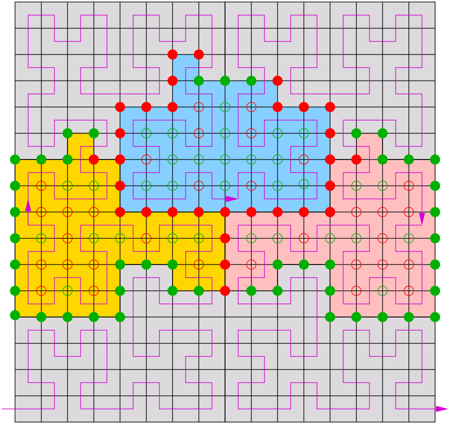
For parallelisation: strategy to determine partitions

1. use a space-filling curve to generate a sequential order on grid cells
2. do a 1D partitioning on the list of cells (cut into equal-sized pieces, or similar)

Possible implementations:

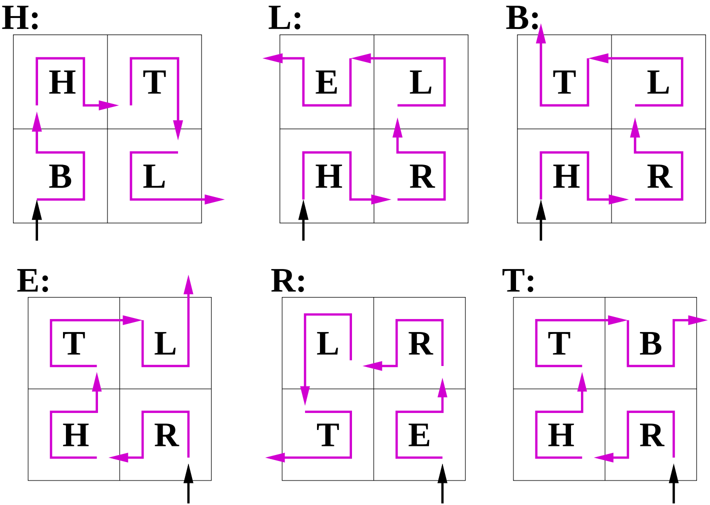
1. compute SFC index (inverse mapping) and sort w.r.t. index
2. modify grammar-based traversals

Hilbert-Curve Partitions on a Cartesian Grid

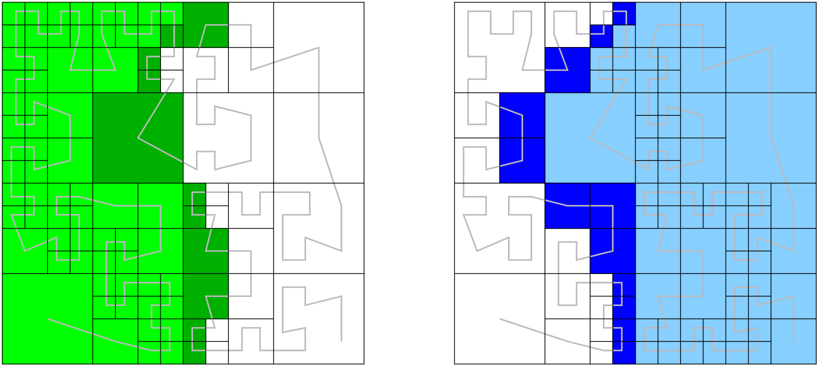


- Hilbert order traversal provides sequential order on grid cells
- Hilbert curve splits vertices into right/left (red/green) set

Turtle Grammar for Left-Right Classification



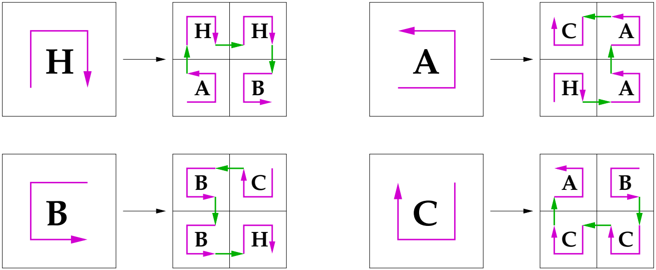
Example: Hilbert-Curve Partitions on Quadtrees



- here: with so-called **ghost cells** (data exchange with neighours) $\rightarrow$ processed in identical order in both partitions

Recall: Grammar to Describe the Hilbert Curve

Construction of the iterations of the Hilbert curve:



Question:

Can this grammar be used to generate **adaptive** Hilbert orders?

Problem:

Words generated by the grammar do not allow reproduction of the refinement info!

A Grammar for Hilbert Orders on Quadtrees

- Non-terminal symbols: $\{H, A, B, C\}$, start symbol H
- terminal characters: $\{\uparrow, \downarrow, \leftarrow, \rightarrow, (,)\}$
- productions:

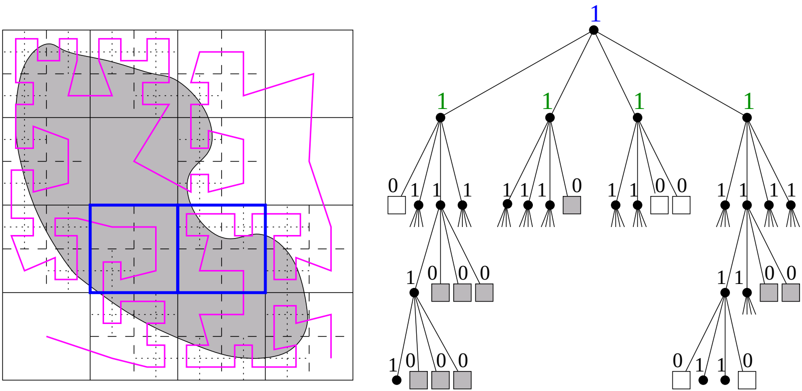
$$
\begin{array}{lcl}
H & \longleftarrow & (A \uparrow H \rightarrow H \downarrow B) \\
A & \longleftarrow & (H \rightarrow A \uparrow A \leftarrow C) \\
B & \longleftarrow & (C \leftarrow B \downarrow B \rightarrow H) \\
C & \longleftarrow & (B \downarrow C \leftarrow C \uparrow A)
\end{array}
$$

$\Rightarrow$ arrows describe the iterations of the Hilbert curve in “turtle graphics”

$\Rightarrow$ terminals (and) mark change of levels: “up” and “down”

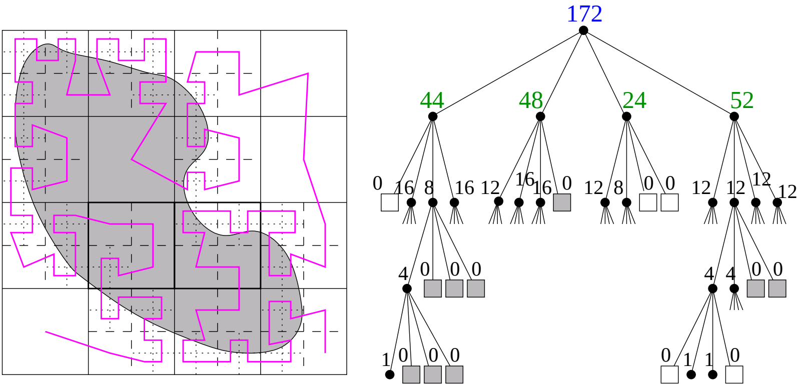
$\Rightarrow$ cmp. algorithm in Python script *sfc_hilbert_plotter_adap*

Hilbert-Order Bitstream-Encoding of a Quadtree



1 1 0 1⋘1 1 1 | 0 0 0 | 0 0 0 | 1⋘ 1 1⋘ 1⋘ 1⋘ | 0 | 1 1⋘ 1⋘ | 0 | 0 | 1 1⋘ 1 1 | 0 | 1 1 | 0 | 1⋘ | 0 0 | 1⋘ 1⋘

Refinement-Tree Encoding of a Quadtree



172	**44**	0	16 ⋘	8	4	1	0 0 0	0 0 0	16 ⋘	**48**	12 ⋘	16 ⋘	16 ⋘	0	**24**	12 ⋘	8 ⋘	0	0	**52**	12 ⋘	12 ⋘	12 ⋘	12 ⋘

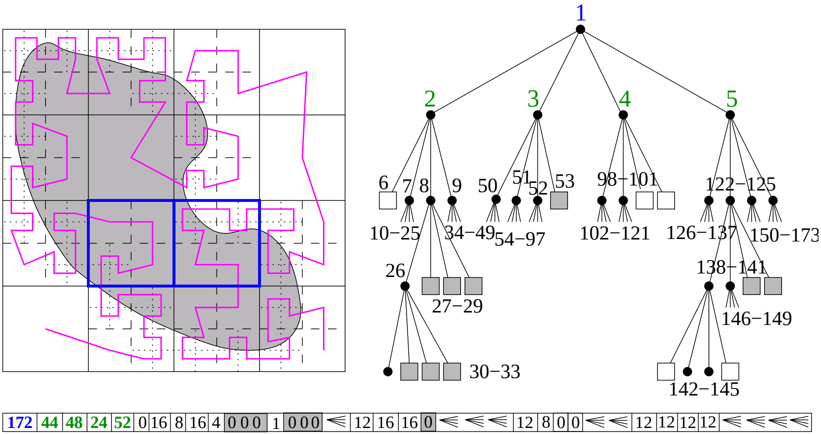
Refinement-Tree Encoding of a Quadtree (2)

REFTREE algorithm for partitioning:

- attributed quadtree with number of leaves/nodes of the subtree for each node
- allows to determine whether a certain node/subtree may be skipped by the current partition (if index of first & last leave/node are given)
- disadvantage of data structure: required information spread across several locations in the stream
  $\Rightarrow$ may be fixed by modified depth-first order

Towards Parallelisation

Refinement-Tree Encoding with Modified Depth-First Order



(numbers in the tree represent position of resp. node information in the stream)

Hölder Continuity of Space-filling Curves

Definition: (*Hölder continuous*)

A function f is called **Hölder continuous with exponent r** on the interval I, if a constant $C > 0$ exists, such that for all $x, y \in I$:

$$\|f(x) - f(y)\|_2 \leq C\,|x - y|^r$$

Importance for space-filling curves:

- $|x - y|$ is the distance of the indices
- $\|f(x) - f(y)\|$ is the distance of the image points (in "space")
- To prove: the Hilbert curve is Hölder coninuous with exponent $r = d^{-1}$, where d is the dimension:

$$\|f(x) - f(y)\|_2 \leq C\,|x - y|^{1/d} = C\,\sqrt[d]{|x - y|}$$

Hölder Continuity of the 3D Hilbert Curve

Proof analogous to simple continuity proof:

- given $x, y \in \mathcal{I}$; find an n, such that $8^{-(n+1)} < |x - y| < 8^{-n}$
- 8^{-n} is the interval lenth for the n-th iteration
  $\Rightarrow [x, y]$ covers at most two neighbouring(!) intervals.
- per construction of the 3D Hilbert curve, the function values $h(x)$ and $h(y)$ are in two adjacent cubes of side length 2^{-n}.
- the length of the space diagonal through the two adjacent cubes is $2^{-n} \cdot \sqrt{1^2 + 1^2 + 2^2} = 2^{-n} \cdot \sqrt{6}$, hence:

$$\begin{aligned} \|h(x) - h(y)\|_2 &\leq 2^{-n}\sqrt{6} = \left(8^{-n}\right)^{1/3}\sqrt{6} = \left(8^{-(n+1)}\right)^{1/3} 8^{1/3}\sqrt{6} \\ &\leq 2\sqrt{6}\,|x - y|^{1/3} \qquad \text{q.e.d.} \end{aligned}$$

Hölder Continuity and Parallelisation

- for the Hilbert curve (also Peano curve and all connected, recursive SFC), we have:

$$\|f(x) - f(y)\|_2 \leq C \sqrt[d]{|x - y|}$$

- relates the distance $|x - y|$ between indices to the disctance $\|f(x) - f(y)\|$ of (mesh) points
- index distance $|x - y|$ equivalent to area covered by the correpsonding curve section $\rightarrow$ **parameterised by area**
- leads to relation between volume (number of grid cells/points) and extent (e.g. radius) of a partition

$\Rightarrow$ Hölder continuity gives a quantitative estimate for **compactness** of partitions