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Sheet 10
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Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
Multi-dimensional Quadrature

In order to find an alternative way to approximate the d-dimensional integral

$$F_d := \int_{[0,1]^d} u(x_1, \ldots, x_d)\, d(x_1, \ldots, x_d)$$

we use a regular d-dimensional grid of step size $h = 2^{-n}$ stored in an array with indices in $[0, 2^n]^d$.

We don't want to be bothered with special boundary treatment so we simplify this task assuming our function is 0 on the boundary, i.e.

$$\forall \mathbf{x} = [x_1, ..., x_d]^T : \exists j \text{ such that } (x_j = 0 \vee x_j = 1) \Rightarrow u(\mathbf{x}) = 0$$

In the next step we compute the d-dimensional hierarchical surpluses:

> **loop** $j = 1, \ldots, d$ **over the dimensions**
>   **loop over all 1d subgrids discretizing spatial direction** j
>   (fix all coordinates except x_j, what you'll get is a 1d array
>   of grid points like on worksheet 4 — for $d = 2$ this comes down
>   to processing all rows and all columns once)
>   **Compute 1d surpluses in dimension** j **(in place) for each subarray**

Once we have the surpluses $v_{\underline{l},\underline{i}}$ (indexing with level $\underline{l}$ and point index $\underline{i}$ as introduced in the lecture) we only need to multiply them with the volume of the associated pagoda and sum everything up. The volume of the pagoda associated with grid point $x_{\underline{l},\underline{i}}$ is $2^{-|\underline{l}|_1}$.

1. Implement this method!
   For $d = 2$ and $n = 3$ this can be done easily in a spreadsheet (using copy and paste is not only ok but also encouraged for thorough understanding).

2. Let

   $$u(x_1, x_2) = 16x_1(1 - x_1)x_2(1 - x_2) = \left(1 - 4(x_1 - \frac{1}{2})^2\right) \cdot \left(1 - 4(x_2 - \frac{1}{2})^2\right).$$

   Write the summed up volumes in descending order (compute with your program or derive from results of worksheet 5). Imagine n to be large.
   How many volumes are at least needed to approximate the integral $(4/9)$ with an absolute error not larger than $1/144$ (this choice is not random, it will give you nice results).

3. Spot and draw the used grid points in the unit square!