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Worksheet 3
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Algorithms of Scientific Computing

Discrete Cosine Transformation

Exercise 1: Two-dimensional Cosine Transformation

The JPEG-method computes the coefficients $\widetilde{F}_{kl}$ from the image data f_{nm} using the following formula

$$\widetilde{F}_{kl} = \frac{1}{N \cdot M} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{nm} \cos\left(\frac{\pi k \left(n+\frac{1}{2}\right)}{N}\right) \cos\left(\frac{\pi l \left(m+\frac{1}{2}\right)}{M}\right)$$

Assume you have a procedure that can compute all coefficients $\widetilde{G}_k$, $k = 0, \ldots, N-1$ efficiently, according to the formula

$$\widetilde{G}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cos\left(\frac{\pi k \left(n+\frac{1}{2}\right)}{N}\right).$$

How can you compute the coefficients $\widetilde{F}_{kl}$ using this procedure?

Exercise 2: Discrete Cosine Transformation

We start with a dataset $f_{-N+1}, \ldots, f_N$, which fulfills the following symmetry constraint:

$$f_{-n} = f_n \qquad \text{for } n = 1, \ldots, N-1$$

a) Show that the corresponding Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \tag{1}$$

are real values only and can be written as:

$$F_k = \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi n k}{N}\right) + \frac{1}{2} f_N \cos(\pi k) \right). \tag{2}$$

b) Show that the F_k is symmetric too.

c) Let `FFT(f,N)` be a procedure that computes the coefficients F_k efficiently (according to equation (1)) from a field `f` which consists of $2N$ values f_n. (The result is written back to field `f`)

Write a short procedure `DCT(g,N)` which uses procedure `FFT` to compute the coefficients F_k for $k = 0, \ldots, N$ from equation (2) for the (non-symmetrical) data $f_0, \ldots, f_N$, stored in the parameter field g.

Exercise 3: Fast Discrete Cosine Transformation

Formulate the butterfly scheme for equation (1) from the previous exercise. Divide the dataset f_n of length $2N$ into a dataset $g_n := f_{2n}$, containing all values with an even index, and a dataset $h_n := f_{2n-1}$, with all values with odd index. Which symmetries can be found in g_n and h_n? Of which kind (Cosine/Sine Transformation, DFT with real data) are the according DFTs of length N? Which symmetries can be found if the dataset f_n fulfills the following symmetry constraint:

$$f_{-n} = f_{n+1}.$$