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Worksheet 4a
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Algorithms of Scientific Computing

Exercise 1: DFT and Least Square Approximation

We set all partial of E derivatives to zero:

$$\sum_{n=-(N-1)/2}^{(N-1)/2} \left[e^{-i2\pi nk/N} \left(f_n - \sum_{p=-(N-1)/2}^{(N-1)/2} \alpha_p e^{i2\pi np/N} \right) \right] = 0. \tag{1}$$

Rearranging the terms gives us the set of N equations

$$\sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk} = \sum_{p=-(N-1)/2}^{(N-1)/2} \alpha_p \sum_{n=-(N-1)/2}^{(N-1)/2} \omega_N^{n(p-k)}, \tag{2}$$

where $\omega_N = e^{i2\pi/N}$.

Next, we will find the second sum in the right hand side of these equations. We notice that N complex numbers ω_N^k, for $k = 0 : N-1$ are the Nth roots of unity because they satisfy

$$(\omega_N^k)^N = \left(e^{i2\pi k/N}\right)^N = e^{i2\pi k} = 1 \tag{3}$$

and therefore are zeros of the polynomial $z^N - 1$. We can factor this polynomial as

$$z^N - 1 = (z-1)\left(z^{N-1} + z^{N-2} + \ldots + z + 1\right) = (z-1)\sum_{n=0}^{N-1} z^n. \tag{4}$$

If $z = \omega_N^{j-k}$, where $j-k$ is not multiple of N, then $z \neq 1$, and we thus have

$$\sum_{n=0}^{N-1} z^n = \sum_{n=0}^{N-1} \omega_N^{(j-k)n} = 0. \tag{5}$$

On the other hand, if $j-k$ is multiple of N then $\omega_N^{j-k} = 1$ and

$$\sum_{n=0}^{N-1} z^n = \sum_{n=0}^{N-1} \omega_N^{(j-k)n} = \sum_{n=0}^{N-1} 1 = N. \tag{6}$$

Since the sequence ω_N^k is N-periodic we get

$$\sum_{n=-(N-1)/2}^{(N-1)/2} \omega_N^{n(p-k)} = N\delta_N(p-k), \tag{7}$$

where $\delta_N(k)$ is known as the modular Kronecker delta. It is defined by

$$\delta_N(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or a multiple of } N, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Using this result in equation (2) we obtain

$$\sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk} = N\alpha_k, \tag{9}$$

for $k = -(N-1)/2 : (N-1)/2$. We get that α_ks correspond exactly to the DFT coefficients

$$\alpha_k = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} f_n \omega_N^{-nk}. \tag{10}$$

Discrete Sine Transformation – Solution optional exercises (not explained during tutorial)

Exercise 2: Fast Discrete Sine Transform

The butterfly scheme is retrieved as usual:

$$\begin{aligned}
F_k &= \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} = \frac{1}{2}\left(\frac{1}{N} \sum_{n=-N/2+1}^{N/2} f_{2n} \omega_{2N}^{-2kn} + \frac{1}{N} \sum_{n=-N/2+1}^{N/2} f_{2n-1} \omega_{2N}^{-k(2n-1)} \right) \\
&= \frac{1}{2}\Bigg(\underbrace{\frac{1}{N} \sum_{n=-N/2+1}^{N/2} f_{2n} \omega_N^{-kn}}_{=:G_k} + \underbrace{\frac{1}{N} \sum_{n=-N/2+1}^{N/2} f_{2n-1} \omega_N^{-kn}}_{=:H_k} \omega_{2N}^{k} \Bigg) \\
&= \frac{1}{2}\left(G_k + \omega_{2N}^k H_k \right) \\
F_{k+N} &= \frac{1}{2}\left(G_{k+N} + \omega_{2N}^{k+N} H_{k+N} \right) = \frac{1}{2}\left(G_k - \omega_{2N}^k H_k \right)
\end{aligned}$$

For the datasets $g_n := f_{2n}$ and $h_n := f_{2n-1}$, respectively, we can try to find other symmetries:

$$g_{-n} = f_{2(-n)} = -f_{-2n} = -f_{2n} = -g_n$$

The "odd" data also shows an odd symmetry and therefore lead to another Sine Transform but with half length.

Analog for the data with odd indices:

$$h_{-n} = f_{2(-n)-1} = f_{-2n-1} = -f_{2n+1} = -f_{2(n+1)-1} = -h_{n+1}$$

Again we get an "odd" symmetry. However, this is the transform shown in the lecture, known as Quarter-Wave-DST, again with half length.

For a dataset with the symmetry constraint $f_{-n} = -f_{n+1}$ we get accordingly

$$g_{-n} = f_{2(-n)} = f_{-2n} = -f_{2n+1} = -h_{n+1}$$

and

$$h_{-n} = f_{-2n-1} = f_{-2n+1} = -f_{2n+2} = -f_{2n-1} = -g_{n+1}$$