

Algorithms of Scientific Computing

Arithmetization of Space-Filling Curves – Solution

Exercise 1: Calculation of h for (in)finite fractions

Initially we calculate the decimal places of the numbers and get:

$$\begin{aligned}\frac{1}{8} &= 0_4.02 \\ \frac{1}{3} &= 0_4.11111111\dots\end{aligned}$$

So, for $h\left(\frac{1}{8}\right)$ we get:

$$\begin{aligned}h\left(\frac{1}{8}\right) &= H_0 \circ H_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = H_0 \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \\ &= H_0 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}\end{aligned}$$

The calculation of $h\left(\frac{1}{3}\right)$ turns out to be much more complicated, since we need to find the following limit:

$$h\left(\frac{1}{3}\right) = h(0_4.1111\dots) = H_1 \circ H_1 \circ \dots \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lim_{n \rightarrow \infty} H_1^n \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, we will write the operator H_1 in the following form:

$$H_1 = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{=:A_1} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}_{=:b_1} = A_1 v + b_1.$$

From this, we get

$$\begin{aligned}H_1^2 v &= A_1(A_1 v + b_1) + b_1 = A_1^2 v + A_1 b_1 + b_1 \\ H_1^3 v &= A_1(A_1^2 v + A_1 b_1 + b_1) + b_1 = A_1^3 v + A_1^2 b_1 + A_1 b_1 + b_1 \\ &\vdots \\ H_1^n v &= A_1^n v + A_1^{n-1} b_1 + \dots + A_1 b_1 + b_1\end{aligned}$$

For the term $A_1^{n-1}b_1 + \dots + A_1b_1 + b_1$ we use a trick, similar to that used for geometric progressions:

$$\begin{aligned} (I - A_1) \left(A_1^{n-1}b_1 + \dots + A_1b_1 + b_1 \right) &= A_1^{n-1}b_1 + \dots + A_1b_1 + b_1 \\ &\quad - A_1^n b_1 - A_1^{n-1}b_1 - \dots - A_1b_1 \\ &= b_1 - A_1^n b_1 = (I - A_1^n) b_1 \end{aligned}$$

This leads to

$$A_1^{n-1}b_1 + \dots + A_1b_1 + b_1 = (I - A_1)^{-1} (I - A_1^n) b_1.$$

which renders to

$$\begin{aligned} \lim_{n \rightarrow \infty} H_1^n &= \lim_{n \rightarrow \infty} \left(A_1^n v + A_1^{n-1}b_1 + \dots + A_1b_1 + b_1 \right) \\ &= \lim_{n \rightarrow \infty} \left((I - A_1)^{-1} (I - \underbrace{A_1^n}_{\rightarrow 0}) b_1 \right) = (I - A_1)^{-1} b_1 \end{aligned}$$

I.e. we get the value $h\left(\frac{1}{3}\right) =: \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ by solving the system of equations

$$(I - A_1) \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = b_1 \quad \Leftrightarrow \quad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

The solution $\xi = 0$ and $\zeta = 1$ can be found quite easily, so finally we get $h\left(\frac{1}{3}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

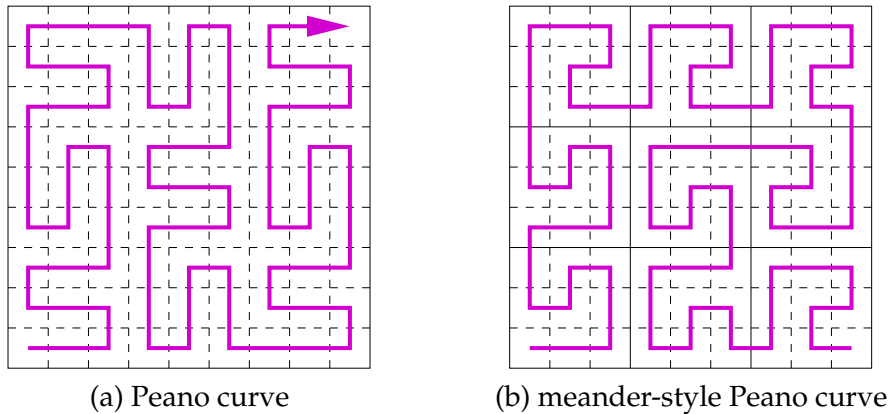


Figure 1: Two space-filling curves of the Peano type

Exercise 2: Arithmetization of the Peano Curve

Like for the arithmetization of the Hilbert curve, we assume that the parameter t is given on the basis 9. $t = 0_9.n_1n_2n_3n_4\dots$. Now we are looking for the operators P_0, \dots, P_8 , so that

$$p(0_9.n_1n_2n_3n_4\dots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \dots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the operators P_i and the algorithm for calculating the Peano functions refer to the Python-script `Worksheet_8.ipynb`.