Algorithms of Scientific Computing

Hierarchical Methods and Sparse Grids
– 1D Hierarchical Basis –

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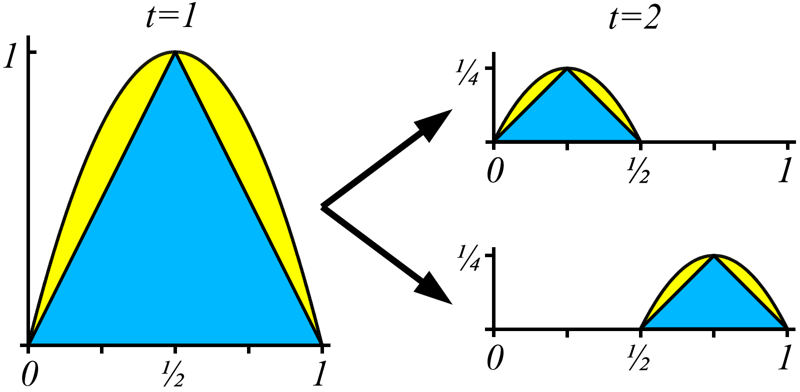
Archimedes' Quadrature

Compute an approximation of $F_1 := \int_0^1 4 \cdot x \cdot (1 - x)\, dx = \frac{2}{3}$



Archimedes' Quadrature (2)

- Integrating $4x(1-x)$, we have to consider several quantities
- Ordered by (recursive) level t:

Level-depth	1	2	3	4	...	t
Mesh-width h	1/2	1/4	1/8	1/16	...	2^{-t}
# triangles	1	2	4	8	...	$\frac{1}{2}2^t$
surplus v	1	1/4	1/16	1/64	...	$4 \cdot 2^{-2t}$
Area of triangle D_1	1/2	1/16	1/128	1/1024	...	$4 \cdot 2^{-3t}$
Sum (current t)	1/2	1/8	1/32	1/128	...	$2 \cdot 2^{-2t}$
Sum ($\leq t$)	1/2	5/8	21/32	85/128	...	$\frac{2}{3}\left(1-2^{-2t}\right)$
Error	1/6	1/24	1/96	1/384	...	$\frac{2}{3}2^{-2t}$

Approximation of Functions

- To analyze Archimedes' quadrature rule, we consider functions
- We need a representation of the (approximating) function $u(x)$ which we are integrating:

  $\rightarrow$ u as linear combination of ansatz functions ϕ_i:

$$u(x) = \sum_{i=1}^{n} \alpha_i \cdot \phi_i(x)$$

- Integrating $u(x)$:

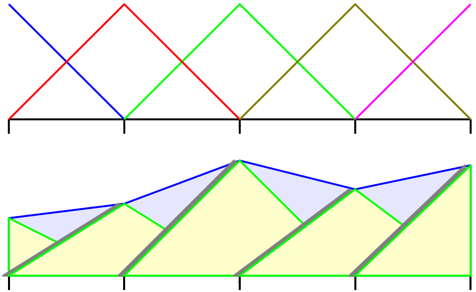
$$\int_a^b u(x)\,dx = \sum_i^n \alpha_i \int_a^b \phi_i(x)\,dx,$$

  - Weighted sum of α_i
  - Remember: Newton-Cotes formulas are weighted sum of function evaluations

Composite Trapezoidal Rule: Function

Interpolant

- Continuous, piecewise linear function
- Represent u in nodal point (hat) basis



- Coefficients α_i are function values at grid points
- Basis functions have area h ($h/2$ at boundaries)

Piecewise Linear Functions

Ansatz space and basis functions

- Only consider $u : [0, 1] \to \mathbb{R}$
- Consider discretization level $n \in \mathbb{N}$
- *Mesh-width* $h_n = 2^{-n}$
- *Grid points* $x_{n,i} = i \cdot h_n$
- Define "mother of all hat functions"

$$\phi(x) := \max\{1 - |x|, 0\}$$

$\Rightarrow$ **Basis functions**

$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- Nodal point basis $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$

Piecewise Linear Functions (2)

- Space of continuous piecewise linear functions

$$V_n = \text{span}\,(\Phi_n)$$

- Interpolants $u_n \in V_n$

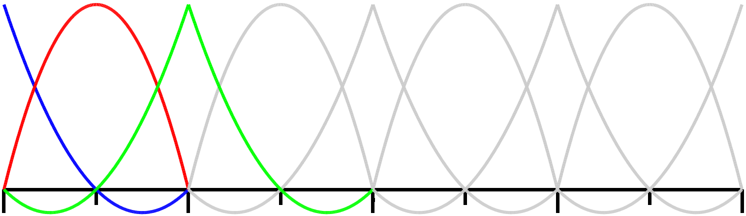
$$u_n(x) = \sum_{i=0}^{2^n} \alpha_{n,i}\phi_{n,i}(x)$$

- V_n the space of all such interpolants u_n

Composite Simpson's Rule: Function

Interplant

- Continuous, piecewise quadratic function
- More complicated basis:



- Basis functions: Lagrangian polynomials, glued together
- α_i: function values at grid points
- Basis functions have area $h/6$ (blue), $4h/6$ (red), $2h/6$ (green)
- We'll not formally define basis functions here...

From Composite Trapezoidal to Archimedes

Piecewise linear functions

- We restrict our functions u to $u(0) = u(1) = 0$
- Nodal point basis for discretization level n:

$$\Phi_n := \{\phi_{n,i}, 1 \leq i \leq 2^n - 1\}$$

- Wanted: *function space*

$$V := \bigcup_{l=1}^{\infty} V_l$$

contains all functions which are in V_l for sufficiently large l

- However: generating system of V as

$$\Phi := \bigcup_{l=1}^{\infty} \Phi_l$$

does not lead to a basis (not linear independent)

Hierarchical Basis

- We are interested in a hierarchical decomposition of V_l

→ Define **hierarchical increment** W_l, such that V_l is a *direct sum*:

$$V_l = V_{l-1} \oplus W_l$$

Side-note: direct sum

- Every $u_l \in V_l$ can be uniquely decomposed as $u_l = u_{l-1} + w_l$, with $u_{l-1} \in V_{l-1}$ and $w_l \in W_l$

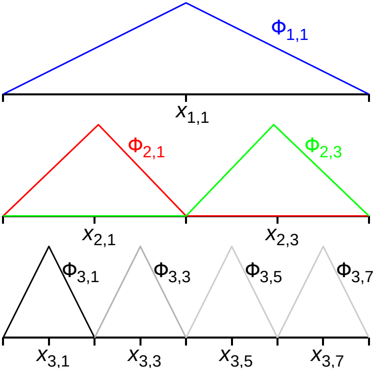
- W_l has to contain 2^{l-1} ansatz functions: $\dim V_l = 2^l - 1 = \dim V_{l-1} + \dim W_l$
- This holds (introducing index sets $\mathcal{I}_l$) for

$$\mathcal{I}_l := \{i : 1 \le i < 2^l,\ i \text{ odd}\}$$

$$W_l := \text{span}\,\{\phi_{l,i} :\ i \in \mathcal{I}_l\}$$

Hierarchical Increments

- Set of hierarchical increments W_l
- For $l = 1$: $W_1 = V_1$
- Example for $l = 1, 2, 3$:



Hierarchical Basis (cont.)

- Then

$$V_n = \bigoplus_{l=1}^{n} W_l$$

is a direct sum, too:

- $u \in V_n$ can be decomposed uniquely into $w_l \in W_l$:

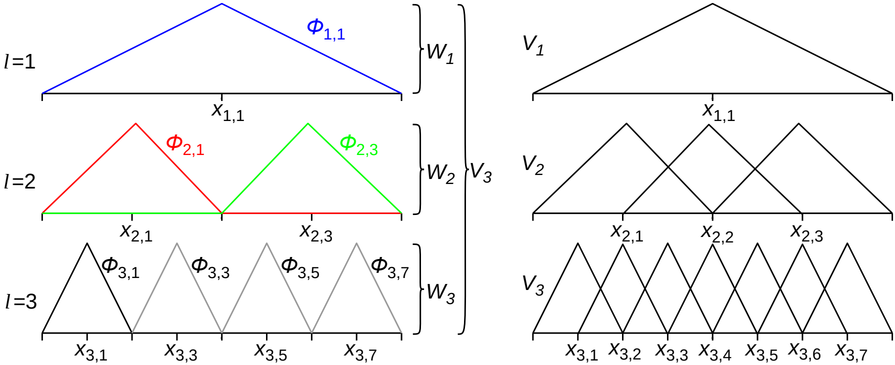
$$u = \sum_{l=1}^{n} w_l = \sum_{l=1}^{n} \sum_{i \in \mathcal{I}_l} v_{l,i} \phi_{l,i}$$

$\rightarrow$ Coefficients $v_{l,i}$ are hierarchical surplusses

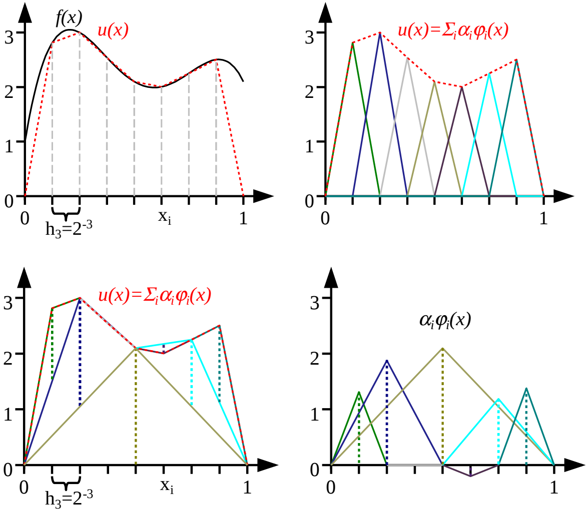
- Corresponding basis of V_n (or, with ∞ instead of n, of V)

$$\Psi_n := \bigcup_{l=1}^{n} \{\phi_{l,i} : i \in \mathcal{I}_l\}.$$

Comparison


l=1
Φ1,1
x1,1
l=2
Φ2,1
Φ2,3
x2,1
x2,3
l=3
Φ3,1
Φ3,3
Φ3,5
Φ3,7
x3,1
x3,3
x3,5
x3,7
W1
W2
W3
V3
V1
V2
V3
x1,1
x2,1
x2,2
x2,3
x3,1
x3,2
x3,3
x3,4
x3,5
x3,6
x3,7

Comparison (2)



Numerical Integration with Hierarchical Basis

– Key Ingredients –

- Integration of $u(x)$:

$$\int_a^b u(x)\,dx = \sum_i^n \alpha_i \int_a^b \phi_i(x)\,dx,$$

- Using a hierarchical basis:

$$\int_a^b u\,dx = \int_a^b \sum_{l=1}^n \sum_{i\in\mathcal{I}_l} v_{l,i}\phi_{l,i}\,dx = \sum_{l=1}^n \sum_{i\in\mathcal{I}_l} v_{l,i} \int_a^b \phi_{l,i}\,dx = \sum_{l=1}^n \sum_{i\in\mathcal{I}_l} v_{l,i} h_l$$

- Computation of hierarchical surpluses:

$$v_{l.i} = u(x_{l,i}) - \tfrac{1}{2}\left(u(x_{l,i-1}) + u(x_{l,i+1})\right)$$

i.e., difference between function and linear interpolant (on coarser level) at $x_{l,i} \to$ **hierarchical surplus**