

Algorithms of Scientific Computing

Space-Filling Curves in 2D and 3D

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Classification of Space-filling Curves

Definition: (*recursive* space-filling curve)

A space-filling curve $f: \mathcal{I} \rightarrow Q \subset \mathbb{R}^n$ is called **recursive**, if both \mathcal{I} and Q can be divided in m subintervals and subdomains, such that

- $f_*(\mathcal{I}^{(\mu)}) = Q^{(\mu)}$ for all $\mu = 1, \dots, m$, and
- all $Q^{(\mu)}$ are geometrically similar to Q .

Definition: (*connected* space-filling curve)

A recursive space-filling curve is called **connected**, if for any two neighbouring intervals $\mathcal{I}^{(\nu)}$ and $\mathcal{I}^{(\mu)}$ also the corresponding subdomains $Q^{(\nu)}$ and $Q^{(\mu)}$ are direct neighbours, i.e. share an $(n - 1)$ -dimensional hyperplane.

Connected, Recursive Space-filling Curves

Examples:

- all Hilbert curves (2D, 3D, ...)
- all Peano curves

Properties: connected, recursive SFC are

- continuous (more exact: Hölder continuous with exponent $1/n$)
- neighbourhood-preserving
- describable by a grammar
- describable in an arithmetic form (similar to that of the Hilbert curve)

Related terms:

- face-connected, edge-connected, node-connected, ...
- also used for the induced orders on grid cells, etc.

Approximating Polygons of the Hilbert Curve

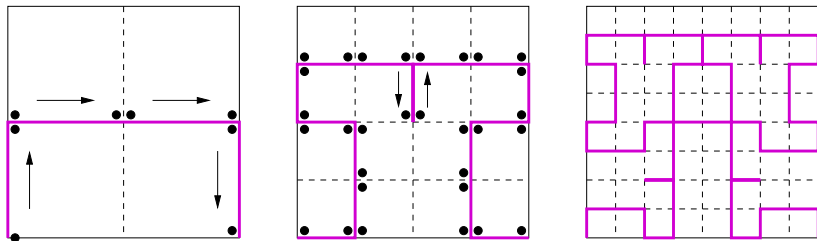
Idea: Connect start and end point of iterate on each subcell.

Definition:

The straight connection of the $4^n + 1$ points

$$h(0), h(1 \cdot 4^{-n}), h(2 \cdot 4^{-n}), \dots, h((4^n - 1) \cdot 4^{-n}), h(1)$$

is called the *n-th approximating polygon of the Hilbert curve*



Properties of the Approximating Polygon

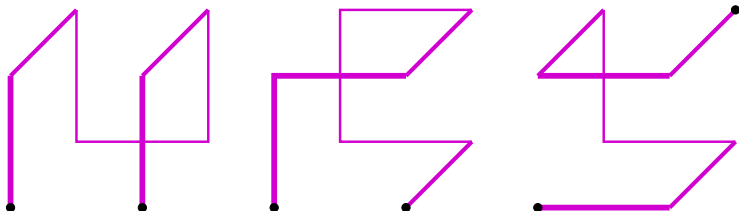
- the approximating Polygon connects the **corners** of the recursively divided subsquares
- the connected corners are start and end points of the space-filling curve within each subsquare
⇒ **assists in the construction of space-filling curves**
- approximating polygons are constructed by recursive repetition of a so-called **Leitmotiv**
⇒ **similarity to Koch and other fractal curves**
- the sequence of corresponding functions $p_n(t)$ converges **uniformly** towards h
⇒ additional proof of continuity of the Hilbert curve

Part I

3D Hilbert Curves

3D Hilbert Curves

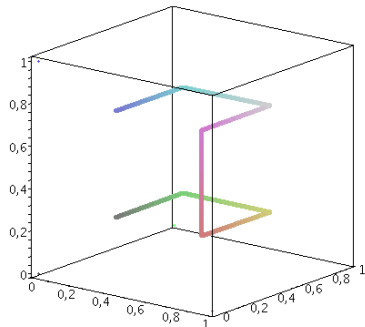
- Wanted: connected, recursive SFC, based on division-by-2
⇒ leads to 3 basic patterns:



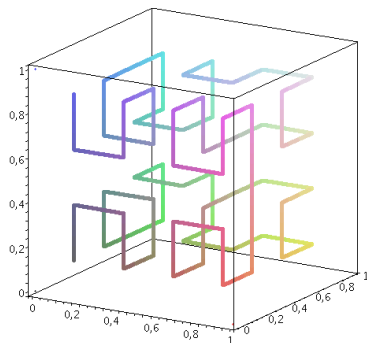
- in addition: symmetric forms, change of orientation
 - always two different orientations of the components
- ⇒ numerous different Hilbert curves expected

Exercise: construct a 3D Hilbert curve!

3D Hilbert Curves – Iterations



1st iteration



2nd iteration

3D Hilbert Curve – Arithmetic Representation

t given in the octal system, $t = 0_8.k_1k_2k_3k_4\dots$, then

$$h(0_8.k_1k_2k_3k_4\dots) = H_{k_1} \circ H_{k_2} \circ H_{k_3} \circ H_{k_4} \circ \dots \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with operators

$$H_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ \frac{1}{2}z + 0 \\ \frac{1}{2}y + 0 \end{pmatrix} \quad H_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + 0 \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}x + 0 \end{pmatrix}$$

$$H_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + 0 \end{pmatrix} \quad H_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}x + \frac{1}{2} \\ -\frac{1}{2}y + \frac{1}{2} \end{pmatrix}$$

3D Hilbert Curve – Arithmetic Representation

(continued)

$$H_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + 1 \\ -\frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix} \quad H_5 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + \frac{1}{2} \end{pmatrix}$$

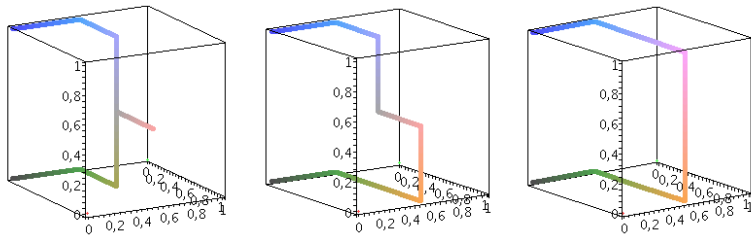
$$H_6 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ -\frac{1}{2}x + 1 \end{pmatrix} \quad H_7 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ -\frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}y + 1 \end{pmatrix}$$

⇒ leads to algorithm analog to 2D Hilbert and 2D Peano

⇒ uses only one pattern; each in only one orientation

3D Hilbert Curves – Variants

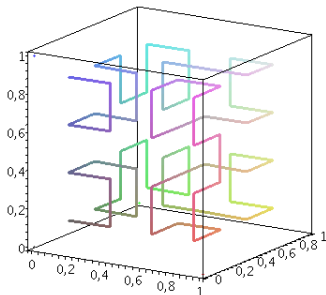
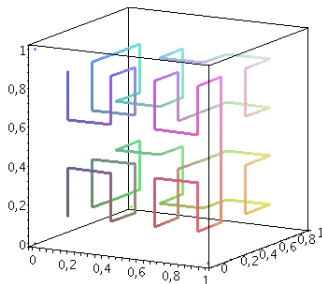
Different approximating polygons:



- same basic pattern:
same order of the eight sub-cubes
- differences only noticeable from the 2nd iteration

3D Hilbert Curves – Variants (2)

Different orientation of the sub-cubes:

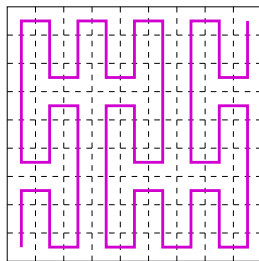
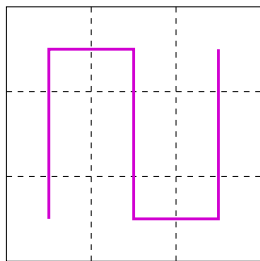


- same basic pattern Grundmotiv, same approximating polygon
- differences only visible from 2nd iteration

Part II

Peano Curves in Higher Dimensions

Construction of the Peano Curve



Recursive Construction:

- divide quadratic domain into 9 subsquares
- construct Peano curve for each subsquare
- join the partial curves to build a higher level curve

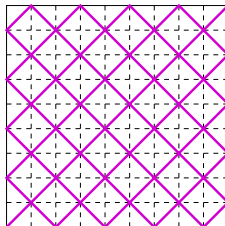
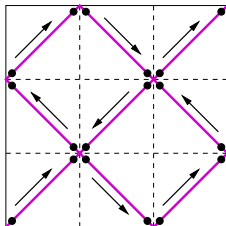
Approximating Polygons of the Peano Curve

Definition:

The straight connection between the $9^n + 1$ points

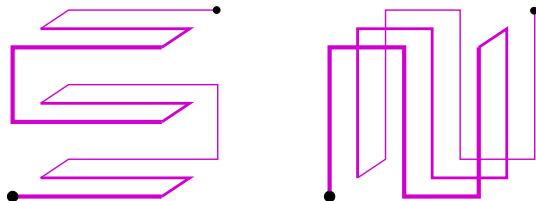
$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \dots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

is called **n -th approximating polygon of the Peano curve**



3D Peano Curves

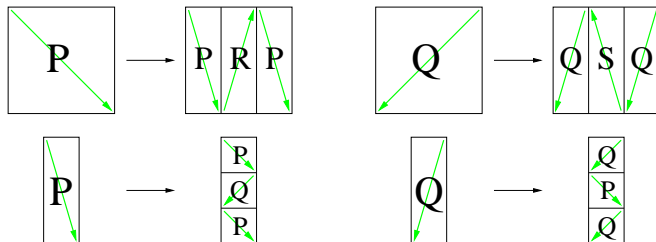
- Concentration on “serpentine” Peano curves (no Meander-type)
- still lots of different variants
- especially interesting are dimension-recursive variants:



in each 3D cut, the sub-cubes are again traversed in Peano order

2D Peano Curve – Dimension-Recursive Grammar

Illustration of patterns:



Construction of Grammar:

$$\begin{array}{l}
 P \longleftarrow P_y \rightarrow R_y \rightarrow P_y \\
 Q \longleftarrow Q_y \leftarrow S_y \leftarrow Q_y
 \end{array}$$

$$\begin{array}{l}
 P_y \longleftarrow P \uparrow Q \uparrow P \\
 Q_y \longleftarrow Q \uparrow P \uparrow Q
 \end{array}$$

Note: dimensional “stretching” implied via index notation (y)

Arithmetic Formulation of the Peano Function

In addition to the classical 2D-construction in the “nonal” system, there is also a dimension-splitting approach based on ternary system:

$t = 0_3.t_1t_2t_3t_4\dots$, then

$$\rho(0_3.t_1t_2t_3t_4\dots) = P_{t_1}^x \circ P_{t_2}^y \circ P_{t_3}^x \circ P_{t_4}^y \circ \dots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

with the operators

$$P_0^x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+0 \\ \frac{1}{3}y+0 \end{pmatrix} \quad P_1^x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+1 \\ \frac{1}{3}y+\frac{1}{3} \end{pmatrix} \quad P_2^x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+0 \\ \frac{1}{3}y+\frac{2}{3} \end{pmatrix}$$

and P^y analogously.

Key idea: each ternary digit defines scaling in only one dimension!

Peano's Representation of the Peano Curve

Definition: (Peano curve, original construction by G. Peano)

- each $t \in \mathcal{I} := [0, 1]$ has a ternary representation

$$t = (0_3.t_1 t_2 t_3 t_4 \dots)$$

- define the mapping $\rho: \mathcal{I} \rightarrow \mathcal{Q} := [0, 1] \times [0, 1]$ as

$$\rho(t) := \begin{pmatrix} 0_3.t_1 k^{t_2}(t_3) k^{t_2+t_4}(t_5) \dots \\ 0_3.k^{t_1}(t_2) k^{t_1+t_3}(t_4) \dots \end{pmatrix}$$

where $k(t_i) := 2 - t_i$ for $t_i = 0, 1, 2$ and k^j is the j -times concatenation of the function k

Peano's Representation of the Peano Curve (2)

Still to prove:

- p is independent of the ternary representation
- the Peano curve $p : \mathcal{I} \rightarrow \mathcal{Q}$ defines a space-filling curve.

Comments:

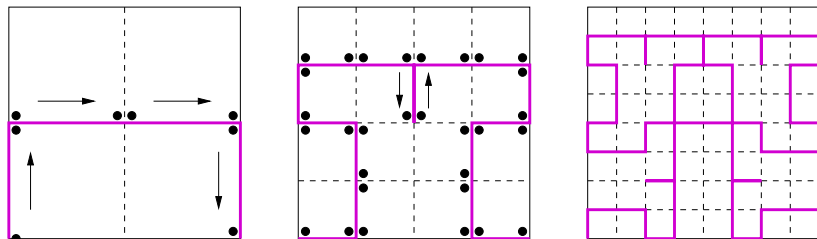
- the direction of “switchback” can be both vertical (see definition), horizontal, or mixed;
- actually, **272 different Peano curves** of the switchback type can be constructed using the same principles;
For comparison: there are only two different 2D Hilbert curves
- in addition: Peano-Meander curves

Part III

Fractal Curves

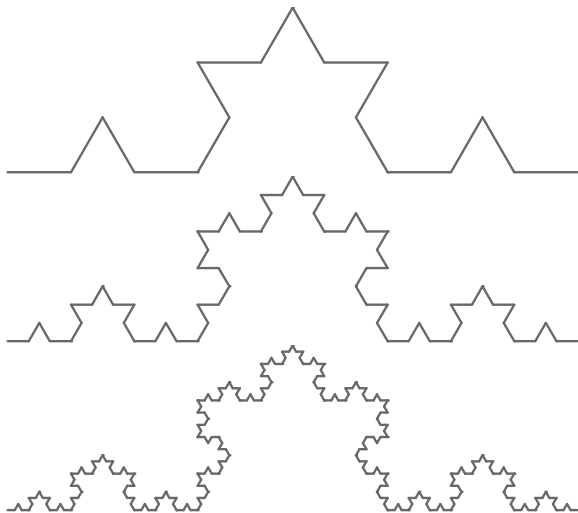
Recall: Approximating Polygons

First approximating polygons of the Hilbert curve:



- polygon results from recursive repetition of a basic pattern
→ “Leitmotiv”
- note: Leitmotiv “added” to alternating sides of the polygon
(compare location of entry/exit points in the illustration)
- strong similarity to **Fractal curves**

Example: Koch Curve



How Long are Approximating Polygons?

Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

⇒ length of the n -th polygon is $2^n \rightarrow \infty$ for $n \rightarrow \infty$.

Corollaries:

- the “length” of the Hilbert curve is not well defined
- instead, we can give an “area” of the Hilbert curve (1, the area of the unit square)

⇒ **Question: what's the dimension of a Hilbert curve?**

Fractal Dimension of Curves

Measuring the length of a curve:

- approx. the curve by a polygon with faces of length ϵ
 \Rightarrow gives a measured length $L(\epsilon)$.
(cmp. approximating polygons of a space-filling curve)
- in case of recursive repeat of a Leitmotiv:
replace each units of length r by a polygon of length q , then

$$L\left(\frac{\epsilon}{r}\right) = \frac{q}{r}L(\epsilon), \quad L(1) := \lambda$$

- we obtain for the length $L(\epsilon)$:

$$L(\epsilon) = \lambda\epsilon^{1-D}, \quad \text{where } D = \log_r q = \frac{\log q}{\log r}$$

Fractal Dimension of Curves (2)

Length of a recursively defined curve computed as

$$L(\epsilon) = \lambda \epsilon^{1-D}, \quad \text{mit } D = \log_r q = \frac{\log q}{\log r}$$

⇒ D is the **fractal dimension** of the curve

⇒ λ is the length w.r.t. that dimension

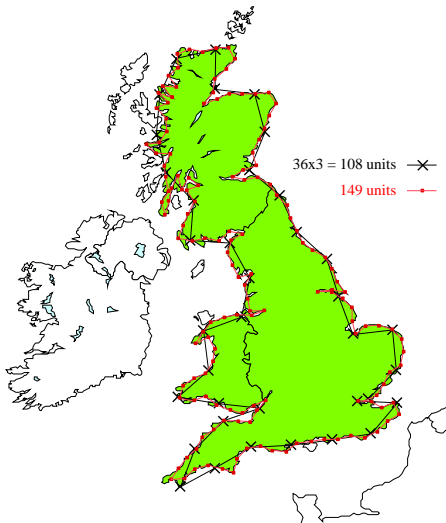
Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or ∞ !
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

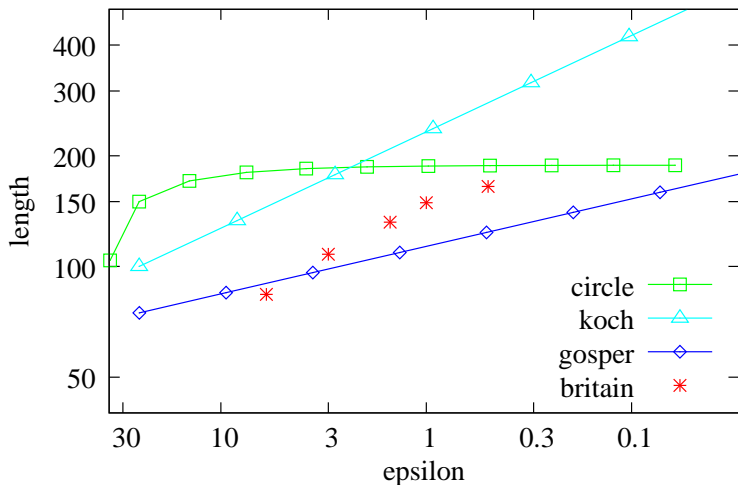
→ **Hausdorff dimension**

How Long is the Coastline of Britain?

Compare, e.g., Mandelbrot: The Fractal Geometry of Nature

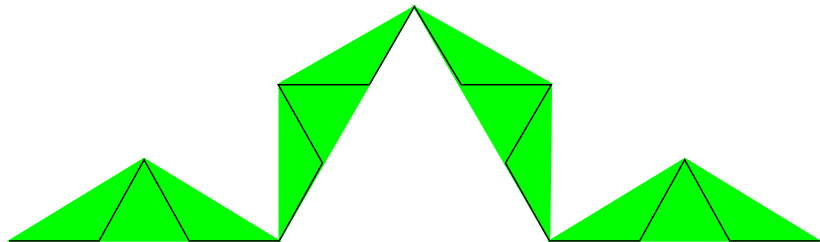


Test: Length of Fractal Curves



Exercise: What is the Area of a Fractal Curve?

Koch curve as example:



→ refine green area and compute its limit value . . .