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SS 2016
Worksheet 11
29.06.2016

Algorithms of Scientific Computing

Arithmetization of Space-Filling Curves and Cache Efficiency

Exercise 1: Arithmetization of the Hilbert Curve

a) Initially we calculate the decimal places of the numbers and get:

$$\begin{aligned}\frac{1}{8} &= 0_4.02\\ \frac{1}{3} &= 0_4.11111111\ldots\end{aligned}$$

So, for $h\left(\frac{1}{8}\right)$ we get:

$$\begin{aligned}h\left(\frac{1}{8}\right) &= H_0 \circ H_2 \begin{pmatrix}0\\0\end{pmatrix} = H_0\left(\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\right)\\ &= H_0\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{4}\\ \frac{1}{4}\end{pmatrix}\end{aligned}$$

The calculation of $h\left(\frac{1}{3}\right)$ turns out to be much more complicated, since we need to find the following limit:

$$h\left(\frac{1}{3}\right) = h(0_4.1111\ldots) = H_1 \circ H_1 \circ \ldots \begin{pmatrix}0\\0\end{pmatrix} = \lim_{n\to\infty} H_1^n \begin{pmatrix}0\\0\end{pmatrix}$$

So, we will write the operator H_1 in matrix-vector form:

$$H_1 = \underbrace{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}}_{=:A_1}\underbrace{\begin{pmatrix}x\\y\end{pmatrix}}_{=:v} + \underbrace{\begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix}}_{=:b_1} = A_1 v + b_1.$$

From this, we get

$$\begin{aligned}H_1^2 v &= A_1(A_1 v + b_1) + b_1 = A_1^2 v + A_1 b_1 + b_1\\ H_1^3 v &= A_1(A_1^2 v + A_1 b_1 + b_1) + b_1 = A_1^3 v + A_1^2 b + A_1 b_1 + b_1\\ \vdots &= \qquad \vdots\\ H_1^n v &= A_1^n v + A_1^{n-1} b_1 + \ldots + A_1 b_1 + b_1\end{aligned}$$

For the term $A_1^{n-1}b_1 + \ldots + A_1b_1 + b_1$ we use a trick, similar to that used for geometric progressions:

$$\begin{aligned}(I - A_1)\left(A_1^{n-1}b_1 + \ldots + A_1b_1 + b_1\right) &= A_1^{n-1}b_1 + \ldots + A_1b_1 + b_1 \\ &\quad -A_1^n b_1 - A_1^{n-1}b_1 - \ldots - A_1b_1 \\ &= b_1 - A_1^n b_1 = (I - A_1^n)\, b_1\end{aligned}$$

This leads to

$$A_1^{n-1}b_1 + \ldots + A_1b_1 + b_1 = (I - A_1)^{-1}(I - A_1^n)\, b_1.$$

which renders to

$$\begin{aligned}\lim_{n\to\infty} H_1^n v &= \lim_{n\to\infty}(A_1^n v + \underbrace{A_1^{n-1}b_1 + \ldots + A_1b_1 + b_1}_{(I-A_1)^{-1}(I-A_1^n)b_1}) \\ &= \lim_{n\to\infty}(\underbrace{A_1^n}_{\to 0} v + (I - A_1)^{-1}(I - \underbrace{A_1^n}_{\to 0})b_1) \\ &= (I - A_1)^{-1}\, b_1\end{aligned}$$

I.e. we get the value $h\left(\frac{1}{3}\right) = \lim_{n\to\infty} H_1^n v = (I - A_1)^{-1}\, b_1$. We can compute the inverse of $(I - A_1)$ easily because it is a diagonal matrix and obtain the final result

$$\begin{aligned}h\left(\frac{1}{3}\right) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\end{aligned}$$

b)

$$\begin{aligned}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} &= h\left(\frac{1}{2}\right) = h\left(\frac{1}{6}\right) = h\left(\frac{5}{6}\right) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= h(0)\end{aligned}$$

Exercise 2: Arithmetization of the Peano Curve

Analogously to the arithmetization of the Hilbert curve, we assume that the parameter t is given on the basis 9. $t = 0_9.n_1n_2n_3n_4\ldots$. Now we are looking for the operators $P_0, \ldots, P_8$, so that

$$p(0_9.n_1n_2n_3n_4\ldots) = P_{n_1} \circ P_{n_2} \circ P_{n_3} \circ P_{n_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

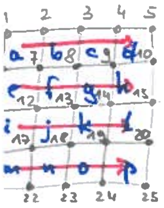
The operators are given as

$$P_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{3}y \end{pmatrix}$$

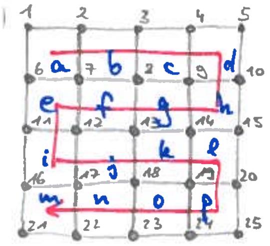
$$
\begin{aligned}
P_1\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{1}{3}x+\frac{1}{3} \\ \frac{1}{3}y+\frac{1}{3} \end{pmatrix} \\
P_2\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{3}y+\frac{2}{3} \end{pmatrix} \\
P_3\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3}x+\frac{1}{3} \\ -\frac{1}{3}y+1 \end{pmatrix} \\
P_4\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{1}{3}x+\frac{2}{3} \\ -\frac{1}{3}y+\frac{2}{3} \end{pmatrix} \\
P_5\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3}x+\frac{1}{3} \\ -\frac{1}{3}y+\frac{1}{3} \end{pmatrix} \\
P_6\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3}x+\frac{2}{3} \\ \frac{1}{3}y \end{pmatrix}. \\
P_7\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{1}{3}x+1 \\ \frac{1}{3}y+\frac{1}{3} \end{pmatrix} \\
P_8\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3}x+\frac{2}{3} \\ \frac{1}{3}y+\frac{2}{3} \end{pmatrix}
\end{aligned}
$$

For the Peano functions see the attached Python program.

x 3

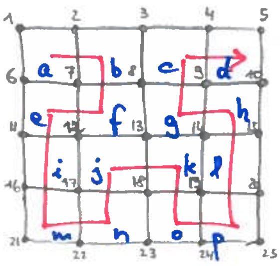


cache status	cell	# cache misses
[1] [2] [6] [7] - - -	a	4
1 [2] 6 [7] [3] [8] -	b	2
[9] 2 6 7 [3] [8] [4]	c	2
[9] [10] [5] 7 3 8 [4]	d	2
9 10 5 [7] [6] [11] [12]	e	3
[8] 10 [13] [7] 6 11 [12]	f	2
[8] [9] [13] 7 [14] 11 12	g	2
8 [9] 13 [15] [14] [10] 12	h	2
[11] [17] [16] 15 14 10 [12]	i	3
11 [17] 16 15 [18] [13] [12]	j	2
[19] 17 16 [14] [18] [13] 12	k	2
[19] 17 [15] [14] 18 13 [20]	l	2
19 [17] 15 [22] [16] [21] 20	m	3
[18] [17] [23] [22] 16 21 20	n	2
[18] 17 [23] 22 [24] 21 [19]	o	2
18 [25] 23 22 [24] [20] [19]	p	2



cache status	cell	# cache misses
[1] [2] [6] [7] - - -	a	4
1 [2] 6 [7] [3] [8] -	b	2
[9] 2 6 7 [3] [8] [4]	c	2
[9] [10] [5] 7 3 8 [4]	d	2
[9] [10] 5 [14] [15] 8 4	h	2
[9] 10 5 [14] 15 [8] 13	g	1
9 [12] [7] 14 15 [8] [13]	f	2
[11] [12] [7] 14 [6] 8 13	e	2
[11] [12] 7 [16] 6 [17] 13	i	2
11 [12] 7 16 [18] [17] [13]	j	1
[19] 12 [14] 16 [18] 17 [13]	k	2
[19] [20] [14] [15] 18 17 13	l	2
[19] [20] 14 15 18 [24] [25]	p	2
[19] 20 [23] 15 [18] [24] 25	o	1
19 [22] [23] [17] [18] 24 25	n	2
[21] [22] 23 [17] 18 24 [16]	m	2

c)



cache status	cell	# cache misses
1 2 6 7	a	4
1 2 6 7 3 8	b	2
13 2 6 7 3 8 12	f	2
13 11 6 7 3 8 12	e	1
13 11 6 7 16 17 12	i	2
21 11 22 7 16 17 12	m	2
21 23 22 18 16 17 12	n	2
21 23 22 18 13 17 12	j	1
14 23 19 18 13 17 12	k	2
14 23 19 18 13 17 24	o	1
14 23 19 18 25 20 24	p	2
14 23 19 15 25 20 24	l	1
14 9 19 15 25 20 10	h	2
14 9 13 15 8 20 10	g	2
14 9 13 15 8 3 4	c	2
14 9 10 5 8 3 4	d	2