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Worksheet 6
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Haar Wavelets

The wavelet families we look at (e.g. Haar wavelets) are constructed around a *mulitresolution analysis*, a nested sequence V_n of function spaces some of which properties are

$$
\begin{aligned}
V_j &\subset V_{j+1},\ j \in \mathbb{Z} && (1)\\
\bigcap_{j=-\infty}^{\infty} V_j &= \{0\} && (2)\\
f(t) \in V_l &\Leftrightarrow f(2^{-l}t) \in V_0 && (3)\\
V_l &= V_{l-1} \oplus W_{l-1}\\
&= V_{l-2} \oplus W_{l-2} \oplus W_{l-1}\\
&= V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_{l-1}, && (4)
\end{aligned}
$$

with *orthogonal* functions $f \in V_j$ and $g \in W_j$, i.e. $< f, g >= 0$.
The theory of multiresolution analysis further states the existence of a unique function ϕ which satisfies a so-called *dilation equation* of the form

$$\phi(t) = \sum_{k\in\mathbb{Z}} c_k \cdot \phi(2t-k). \tag{5}$$

Define another function, known as the **mother wavelet** or the **wavelet function** of the form

$$\psi(t) := \sum_{k\in\mathbb{Z}} (-1)^k c_{1-k} \cdot \phi(2t-k). \tag{6}$$

With the help of ϕ and ψ, we can define *orthonormal nodal bases* $\{ \phi_{l,k} \}$ for V_l with

$$
\begin{aligned}
\phi_{l,k}(t) &= \phi(2^l\, t-k)\\
span\{ \phi_{l,k} \} &= V_l,\ < \phi_{l,k}, \phi_{l,m} >= \delta_{k,m}\ \ k,m \in \mathbb{Z}. && (7)
\end{aligned}
$$

The function ϕ is called **father wavelet** or the **scaling function**, and together with a **mother wavelet** ψ, they define the wavelet family. It is not necessary to know a specific formula for ϕ, the dilation equation (5) with its coefficients c_k together with the theory of multiresolution analysis provide enough information to derive the mother wavelet ψ as well as *orthonormal wavelet bases* $\{ \psi_{l,m} \}$ for the W_l with

$$
\begin{aligned}
\psi_{l,k}(t) &= \psi(2^l\, t-k)\\
span\{ \psi_{l,k} \} &= W_l,\ < \psi_{l,k}, \psi_{l,m} >= \delta_{k,m}\ \ k,m \in \mathbb{Z}. && (8)
\end{aligned}
$$

Excercise 1: Cranking the Machine

Typically the scaling function ϕ is not known explicitly, and sometimes a closed-form analytic formula does not even exist. However, for continuous ϕ we can approximate the function to arbitrarily high precision using the "Cascade Algorithm", a fixed-point method for functions.

In this exercise we want to implement this algorithm by iterating over the expression

$$F(\gamma)(t) = \sum_k c_k \cdot \gamma(2t-k) \tag{9}$$

in order to find the fixed point γ of F. That is, at iteration n

$$\gamma_{n+1}(t) = \sum_k c_k \cdot \gamma_n(2t-k) \tag{10}$$

Our starting point γ_0 will be the hat function

$$\gamma_0(t) = \max\{1-|x|, 0\}. \tag{11}$$

(i) Over the interval $[-1,4]$ plot the approximations of the scaling function ϕ for the Haar wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients $c_k,\ k = 0,1$ in (12) into (9) resp. (5).

$$c_0 = c_1 = 1 \tag{12}$$

(ii) Over the interval $[-1,4]$ plot the approximations of the scaling function ϕ for the Daubechies wavelet family obtained in the first 7 iterations of the cascade algorithm. Do so by plugging the refinements coefficients $c_k,\ k = 0,\dots,3$ in (13) into (9) resp. (5).

$$c_0 = \frac{1+\sqrt{3}}{4} \qquad c_1 = \frac{3+\sqrt{3}}{4} \qquad c_2 = \frac{3-\sqrt{3}}{4} \qquad c_3 = \frac{1-\sqrt{3}}{4} \tag{13}$$

Excercise 2: The Haar Wavelet Basis

We derive the *mother wavelet* ψ as well as *orthonormal wavelet bases* $\{\ \psi_{l,m}\ \}$ with

$$\begin{aligned} \psi_{l,k}(t) &= \psi(2^l\, t - k) \\ span\{\ \psi_{l,k}\ \} &= W_l,\ < \psi_{l,k}, \psi_{l,m} >= \delta_{k,m}\ \ k,m \in \mathbb{Z}. \end{aligned} \tag{14}$$

In this exercise we want to compute the 1-d wavelet transform for the Haar wavelet family and apply it to a signal vector $\vec{s}$ of length $m = 2^n$. The transform can be implemented very efficiently as a "pyramidal algorithm" taking $\mathcal{O}(m)$ steps. For educational purpose we focus on the $\mathcal{O}(m^2)$ matrix-based algorithm.

(i) Write a function that constructs the transformation matrix M consisting of the basis vectors $\psi_{l,k},\ l \le n,\ 0 \le k \le 2^n - 1$.

(ii) Use Python's package *numpy.linalg* to invert the matrix.

(iii) Use the program to compute the transform $\vec{d} = M^{-1}\vec{s}$ as well as the reconstructed signal $\vec{s} = M\vec{d}$ of the vector

$$\vec{s} = [1,2,3,-1,1,-4,-2,4]^T$$

(iv) Verify the program's output tracing the steps by hand.