Exercise 1: Function approximation

In problems of function interpolation or regression, we are given a set of data points \( \{(y_1, x_1), \ldots, (y_m, x_m)\} \), we seek for a function \( f \in \mathbb{V} \), such that the distance between this function and all data points are minimized, e.g.,

\[
    f(x) = \arg\min_f \left\{ \sum_{j=1}^{m} \|y_j - f(x_j)\| \right\}.
\]

We can express \( f \) in the form of

\[
    f(x) = \sum_i \alpha_i \cdot \phi_i(x),
\]

where \( \{\phi_i\} \) is a set of known basis functions that span the function space, i.e., \( \mathbb{V} = \text{span}\{\phi_i\} \). Now the problem of searching for \( f \) has become searching for \( \{\alpha_i\} \), the coefficients of basis functions.

In this exercise, we would like to employ the same strategy to approximate a function \( g(x) \). The goal is to look for an \( f \) of the form (2) that can best approximate \( g(x) \), e.g.,

\[
    f(x) = \arg\min_f \|g(x) - f(x)\|_2.
\]

1. Show that solving (3) is equivalent to finding a set of coefficients \( \{\alpha_k\} \), where

\[
    \int_{-\infty}^{+\infty} \phi_k(x) \left( g(x) - \sum_i \alpha_i \cdot \phi_i(x) \right) dx = 0, \forall k
\]
Proof:

\[ f(x) = \arg \min_f \|g(x) - f(x)\|_2 \]

\[ = \arg \min_f \left\{ \int_{\Omega} (g(x) - f(x))^2 \, dx \right\} \]

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\[ \Leftrightarrow \alpha = \arg \min_{\alpha} \left\{ \int_{\Omega} \left( g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \, dx \right\} \]

Let \( H(\alpha) = \int_{\Omega} \left( g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \, dx \). Since \( H(\alpha) \geq 0 \),

\[ \alpha = \arg \min_{\alpha} \{ H(\alpha) \} \]

is equivalent to finding \( \alpha \), such that

\[ \frac{\partial}{\partial \alpha_k} H(\alpha) = 0, \forall k \]

\[ \int_{\Omega} \left( g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \, dx = 0, \forall k \]

\[ \int_{\Omega} \frac{\partial}{\partial \alpha_k} \left( g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \, dx = 0, \forall k \]

\[ \int_{\Omega} 2 \left( g(x) - \sum_i \alpha_i \phi_i(x) \right) \phi_k(x) \, dx = 0, \forall k \]

\[ \int_{\Omega} \phi_k(x) \left( g(x) - \sum_i \alpha_i \phi_i(x) \right) \, dx = 0, \forall k \]

2. Transform (4) into a linear system of the form \( A\alpha = b \). What does \( A \) look like?

Solution:

\[ \int_{\Omega} \phi_k(x) \left( g(x) - \sum_i \alpha_i \phi_i(x) \right) \, dx = 0, \forall k \]

\[ \int_{\Omega} \left( \phi_k(x) g(x) - \sum_i \alpha_i \phi_k(x) \phi_i(x) \right) \, dx = 0, \forall k \]

\[ \int_{\Omega} \phi_k(x) g(x) \, dx - \int_{\Omega} \sum_i \alpha_i \phi_k(x) \phi_i(x) \, dx = 0, \forall k \]
Therefore,
\[
\sum_i \left( \int_{\Omega} \phi_k(x) \phi_i(x) \, dx \right) \alpha_i = \int_{\Omega} \phi_k(x) g(x) \, dx, \forall k.
\] (5)

Written in matrix form is
\[
A \alpha = b \quad (6)
\]
where \(A\) is a square, symmetric and banded matrix with entries \(a_{ki} = \int_{\Omega} \phi_k(x) \phi_i(x) \, dx\). \(a_{ki} \neq 0\) when \(\phi_k(x)\) and \(\phi_i(x)\) overlap, 0 otherwise. \(A\) will be different depending on the choice the basis functions.

3. Let \(g(x) = -4x^2 + 4x, \) where \(x \in [0, 1] \). Solve (4) with nodal basis hat functions and piecewise constant functions.

**Solution:** Choose a discretization level, e.g., \(h = 2^{-l}, \) \(l = 3\) (number of basis supports are fixed). Substitude \(g(x)\) and the basis functions into (5), get a linear system, solve.

### Exercise 2: Discrete Wavelet Transform

Compute the DWT for the Haar wavelets for the signal \(s = [8, 4, -1, 1, 0, 4, 1, 7, -\frac{5}{2}, -\frac{3}{2}, 0, -4, -2, -2, 1, -5]\) using the Pyramidal Algorithm. Discuss the computation complexity of this method.

**Solution:**

Let \(H_l\) be the high-pass filters and \(L_l\) the low-pass filters. \(d\) denote the output vector.

**Step 1:**
\[
d_3 = H_4 \cdot s = [2, -1, -2, -3, -\frac{1}{2}, 2, 0, 3], \text{ and } \\
c_3 = L_4 \cdot s = [6, 0, 2, 4, -2, -2, -2, -2]
\]

**Step 2:**
\[
d_2 = H_3 \cdot c_3 = [3, -1, 0, 0], \text{ and } \\
c_2 = L_3 \cdot c_3 = [3, 3, -2, -2]
\]

**Step 3:**
\[
d_1 = H_2 \cdot c_2 = [0, 0], \text{ and } \\
c_1 = L_2 \cdot c_2 = [3, -2]
\]

**Step 3:**
\[
d_0 = H_1 \cdot c_1 = \left[\frac{5}{2}\right], \text{ and } \\
c_0 = L_1 \cdot c_1 = \left[\frac{1}{2}\right]
\]

\[
d = [c_0, d_0, d_1, d_2, d_3] = \left[\frac{5}{2}, \frac{5}{2}, 0, 0, 3, -1, 0, 0, 2, -1, -2, -3, -\frac{5}{2}, 2, 0, 3\right]
\]

### Exercise 3: Discrete Wavelet Transform 2D

Compute the DWT for the Haar wavelets for the 2D signal \(s = \begin{bmatrix} 4 & 2 & 3 & 5 \\ 1 & -7 & 0 & 8 \\ -1 & -3 & 9 & -3 \\ 6 & -2 & -1 & 1 \end{bmatrix}\).