Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
M.Sc. A. Mo-Hellenbrand, M.Sc. A. Schwarz

SS 2016
Worksheet 7
01.6.2016

Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Wavelets and Finite Element Method

Exercise 1: Function approximation

In problems of function interpolation or regression, we are given a set of data points $\{(y_1,x_1),\ldots,(y_m,x_m)\}$, we seek for a function $f \in \mathbb{V}$, such that the distance between this function and all data points are minimized, e.g.,

$$f(x) = \arg\min_f \left\{ \sum_{j=1}^{m} \|y_j - f(x_j)\| \right\}. \tag{1}$$

We can express f in the form of

$$f(x) = \sum_i \alpha_i \cdot \phi_i(x), \tag{2}$$

where $\{\phi_i\}$ is a set of known basis functions that span the function space, i.e., $\mathbb{V} = \mathrm{span}\{\phi_i\}$. Now the problem of searching for f has become searching for $\{\alpha_i\}$, the coefficients of basis functions.

In this exercise, we would like to employ the same strategy to approximate a function $g(x)$. The goal is to look for an f of the form (2) that can best approximate $g(x)$, e.g.,

$$f(x) = \arg\min_f \|g(x) - f(x)\|_2. \tag{3}$$

1. Show that solving (3) is equivalent to finding a set of coefficients $\{\alpha_k\}$, where

$$\int_{-\infty}^{+\infty} \phi_k(x) \left(g(x) - \sum_i \alpha_i \cdot \phi_i(x) \right) \mathbf{d}x \stackrel{!}{=} 0,\ \forall k \tag{4}$$

Proof:

$$
\begin{aligned}
f(x) &= \arg\min_f \|g(x) - f(x)\|_2 \\
&= \arg\min_f \left\{ \sqrt{\int_\Omega (g(x) - f(x))^2 \, \mathbf{d}x} \right\} \\
&= \arg\min_f \left\{ \int_\Omega (g(x) - f(x))^2 \, \mathbf{d}x \right\} \\
\Leftrightarrow \; \alpha &= \arg\min_\alpha \left\{ \int_\Omega \left(g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \mathbf{d}x \right\}
\end{aligned}
$$

Let $H(\alpha) = \int_\Omega \left(g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \mathbf{d}x$. Since $H(\alpha) \geq 0$,

$$
\alpha = \arg\min_\alpha \{H(\alpha)\}
$$

is equivalent to finding α, such that

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_k} H(\alpha) &\stackrel{!}{=} 0, \; \forall k \\
\frac{\partial}{\partial \alpha_k} \left[\int_\Omega \left(g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \mathbf{d}x \right] &\stackrel{!}{=} 0, \; \forall k \\
\int_\Omega \left[\frac{\partial}{\partial \alpha_k} \left(g(x) - \sum_i \alpha_i \phi_i(x) \right)^2 \right] \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k \\
\int_\Omega 2 \left(g(x) - \sum_i \alpha_i \phi_i(x) \right) \phi_k(x) \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k \\
\int_\Omega \phi_k(x) \left(g(x) - \sum_i \alpha_i \phi_i(x) \right) \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k
\end{aligned}
$$

2. Transform (4) into a linear system of the form $A\alpha = b$. What does A look like?

Solution:

$$
\begin{aligned}
\int_\Omega \phi_k(x) \left(g(x) - \sum_i \alpha_i \phi_i(x) \right) \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k \\
\int_\Omega \left(\phi_k(x) g(x) - \sum_i \alpha_i \phi_k(x) \phi_i(x) \right) \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k \\
\int_\Omega \phi_k(x) g(x) \mathbf{d}x - \int_\Omega \sum_i \alpha_i \phi_k(x) \phi_i(x) \mathbf{d}x &\stackrel{!}{=} 0, \; \forall k
\end{aligned}
$$

Therefore,

$$\sum_i \left(\int_\Omega \phi_k(x)\phi_i(x)\mathbf{d}x \right) \alpha_i = \int_\Omega \phi_k(x) g(x)\mathbf{d}x,\ \forall k. \tag{5}$$

Written in matrix form is

$$A\alpha = b \tag{6}$$

where A is a square, symmetric and banded matrix with entries $\{a_{ki} = \int_\Omega \phi_k(x)\phi_i(x)\mathbf{d}x\}$. $a_{ki} \neq 0$ when $\phi_k(x)$ and $\phi_i(x)$ overlap, 0 otherwise. A will be different depending on the choice the basis functions.

3. Let $g(x) = -4x^2 + 4x$, where $x \in [0,1]$. Solve (4) with nodal basis hat functions and piecewise constant functions.

Solution: Choose a discritization level, e.g., $h = 2^{-l}$, $l = 3$ (number of basis supports are fixed). Substitude $g(x)$ and the basis functions into (5), get a linear system, solve.

Exercise 2: Discrete Wavelet Transform

Compute the DWT for the Haar wavelets for the signal $s = [8,4,-1,1,0,4,1,7,-\frac{5}{2},-\frac{3}{2},0,-4,-2,-2,1,-5]$ using the Pyramidal Algorithm. Discuss the computation complexity of this method.

Solution:

Let H_l be the high-pass filters and L_l the low-pass filters. d denote the output vector.

Step 1: $d_3 = H_4 \cdot s = [2,-1,-2,-3,-\frac{1}{2},2,0,3]$, and
$c_3 = L_4 \cdot s = [6,0,2,4,-2,-2,-2,-2]$

Step 2: $d_2 = H_3 \cdot c_3 = [3,-1,0,0]$, and
$c_2 = L_3 \cdot c_3 = [3,3,-2,-2]$

Step 3: $d_1 = H_2 \cdot c_2 = [0,0]$, and
$c_1 = L_2 \cdot c_2 = [3,-2]$

Step 3: $d_0 = H_1 \cdot c_1 = [\frac{5}{2}]$, and
$c_0 = L_1 \cdot c_1 = [\frac{1}{2}]$

$d = [c_0,d_0,d_1,d_2,d_3] = [\frac{1}{2},\frac{5}{2},0,0,3,-1,0,0,2,-1,-2,-3,-\frac{1}{2},2,0,3]$

Exercise 3: Discrete Wavelet Transform 2D

Compute the DWT for the Haar wavelets for the 2D signal $s = \begin{bmatrix} 4 & 2 & 3 & 5 \\ 1 & -7 & 0 & 8 \\ -1 & -3 & 9 & -3 \\ 6 & -2 & -1 & 1 \end{bmatrix}$.