Algorithms for Scientific Computing

Space-Filling Curves in 2D and 3D

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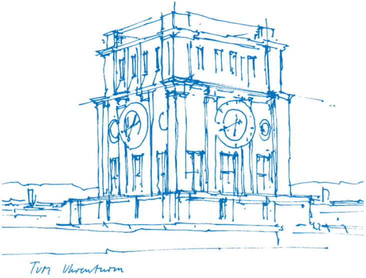


Classification of Space-filling Curves

Definition: (*recursive* space-filling curve)

A space-filling curve $f\colon \mathcal{I} \to \mathcal{Q} \subset \mathbb{R}^n$ is called **recursive**, if both $\mathcal{I}$ and $\mathcal{Q}$ can be divided in m subintervals and sudomains, such that

- $f_*(\mathcal{I}^{(\mu)}) = \mathcal{Q}^{(\mu)}$ for all $\mu = 1, \ldots, m$, and
- all $\mathcal{Q}^{(\mu)}$ are geometrically similar to $\mathcal{Q}$.

Definition: (*connected* space-filling curve)

A recursive space-filling curve is called **connected**, if for any two neighbouring intervals $\mathcal{I}^{(\nu)}$ and $\mathcal{I}^{(\mu)}$ also the corresponding subdomains $\mathcal{Q}^{(\nu)}$ and $\mathcal{Q}^{(\mu)}$ are direct neighbours, i.e. share an $(n-1)$-dimensional hyperplane.

Connected, Recursive Space-filling Curves

Examples:

- all Hilbert curves (2D, 3D, ...)
- all Peano curves

Properties: connected, recursive SFC are

- continuous (more exact: Hölder continuous with exponent $1/n$)
- neighbourship-preserving
- describable by a grammar
- describable in an arithmetic form
  (similar to that of the Hilbert curve)

Related terms:

- face-connected, edge-connected, node-connected, ...
- also used for the induced orders on grid cells, etc.

Approximating Polygons of the Hilbert Curve

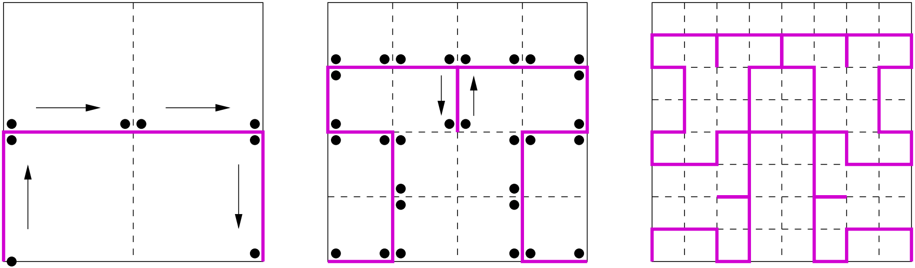
Idea: Connect start and end point of iterate on each subcell.

Definition:

The straight connection of the $4^n + 1$ points

$$h(0), h(1 \cdot 4^{-n}), h(2 \cdot 4^{-n}), \ldots, h((4^n - 1) \cdot 4^{-n}), h(1)$$

is called the *n-th approximating polygon of the Hilbert curve*



Properties of the Approximating Polygon

- the approximating Polygon connects the **corners** of the recursively divided subsquares
- the connected corners are start and end points of the space-filling curve within each subsquare

  $\Rightarrow$ **assists in the construction of space-filling curves**

- approximating polygons are constructed by recursive repetition of a so-called **Leitmotiv**

  $\Rightarrow$ **similarity to Koch and other fractal curves**

- the sequence of corresponding functions $p_n(t)$ converges **uniformly** towards h

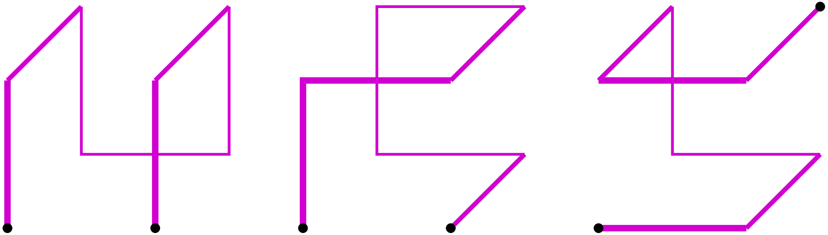
  $\Rightarrow$ additional proof of continuity of the Hilbert curve

Part I

3D Hilbert Curves

3D Hilbert Curves

- Wanted: connected, recursive SFC, based on division-by-2
  ⇒ leads to 3 basic patterns:

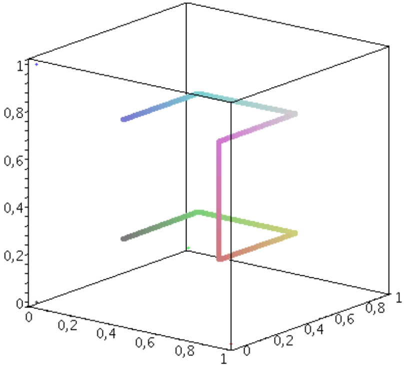


- in addition: symmetric forms, change of orientation
- always two different orientations of the components

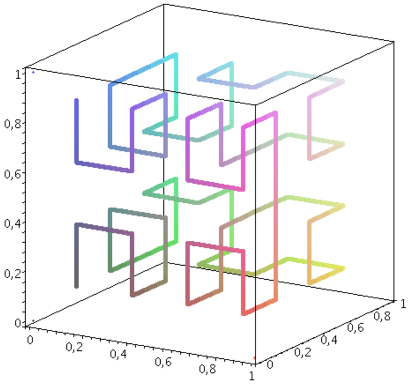
⇒ numerous different Hilbert curves expected

Exercise: construct a 3D Hilbert curve!

3D Hilbert Curves – Iterations



1st iteration



2nd iteration

3D Hilbert Curve – Arithmetic Representation

t given in the octal system, $t = 0_8.k_1k_2k_3k_4\ldots$, then

$$h(0_8.k_1k_2k_3k_4\ldots) = H_{k_1} \circ H_{k_2} \circ H_{k_3} \circ H_{k_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with operators

$$H_0\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ \frac{1}{2}z + 0 \\ \frac{1}{2}y + 0 \end{pmatrix} \quad H_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + 0 \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}x + 0 \end{pmatrix}$$

$$H_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + 0 \end{pmatrix} \quad H_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}x + \frac{1}{2} \\ -\frac{1}{2}y + \frac{1}{2} \end{pmatrix}$$

3D Hilbert Curve – Arithmetic Representation

(continued)

$$H_4\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + 1 \\ -\frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix} \quad H_5\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ \frac{1}{2}z + \frac{1}{2} \end{pmatrix}$$

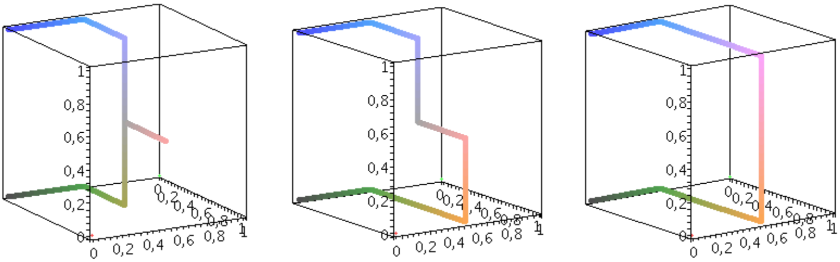
$$H_6\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} \\ -\frac{1}{2}x + 1 \end{pmatrix} \quad H_7\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + 0 \\ -\frac{1}{2}z + \frac{1}{2} \\ -\frac{1}{2}y + 1 \end{pmatrix}$$

$\Rightarrow$ leads to algorithm analog to $2D$ Hilbert and $2D$ Peano

$\Rightarrow$ uses only one pattern; each in only one orientation

3D Hilbert Curves – Variants

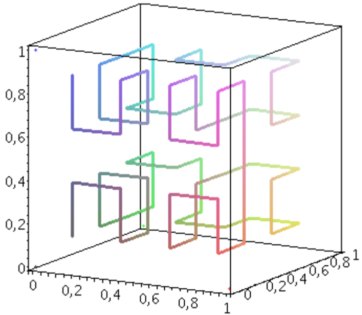
Different approximating polygons:

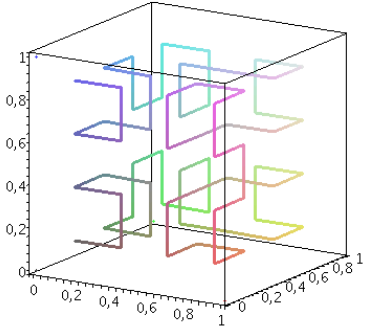


- same basic pattern:
  same order of the eight sub-cubes
- differences only noticeable from the 2nd iteration

3D Hilbert Curves – Variants (2)

Different orientation of the sub-cubes:



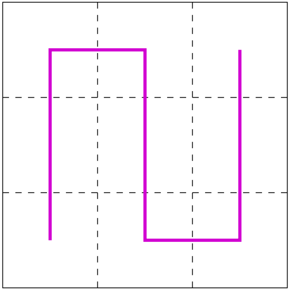


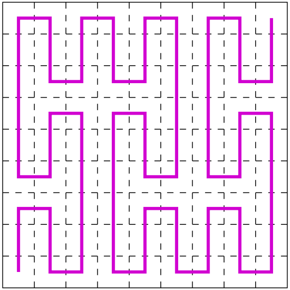
- same basic pattern Grundmotiv, same approximating polygon
- differences only visible from 2nd iteration

Part II

Peano Curves in Higher Dimensions

Construction of the Peano Curve





Recursive Construction:

- divide quadratic domain into 9 subsquares
- construct Peano curve for each subsquare
- join the partial curves to build a higher level curve

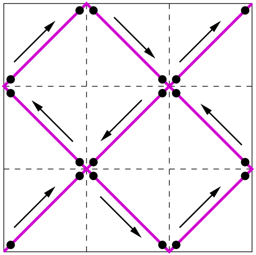
Approximating Polygons of the Peano Curve

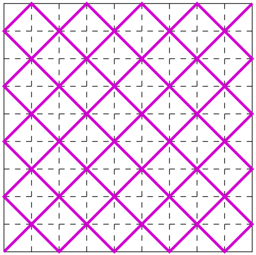
Definition:

The straight connection between the $9^n + 1$ points

$$p(0), p(1 \cdot 9^{-n}), p(2 \cdot 9^{-n}), \ldots, p((9^n - 1) \cdot 9^{-n}), p(1)$$

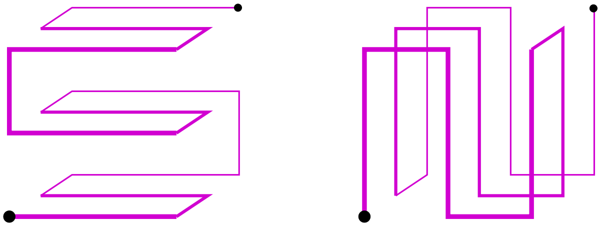
is called ***n*-th approximating polygon of the Peano curve**





3D Peano Curves

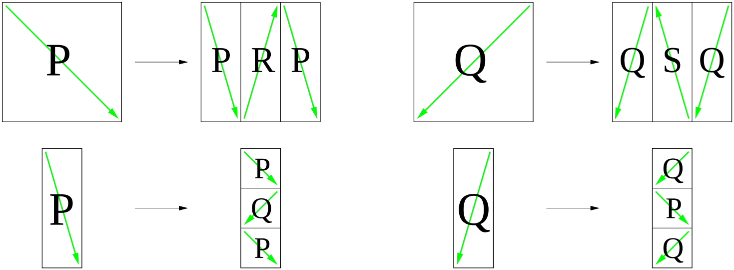
- Concentration on “serpentine” Peano curves (no Meander-type)
- still lots of different variants
- especially interesting are dimension-recursive variants:



in each 3D cut, the sub-cubes are again traversed in Peano order

2D Peano Curve – Dimension-Recursive Grammar

Illustration of patterns:



Construction of Grammar:

$$
\begin{array}{lcl}
P & \longleftarrow & P_y \rightarrow R_y \rightarrow P_y \\
Q & \longleftarrow & Q_y \leftarrow S_y \leftarrow Q_y
\end{array}
\qquad
\begin{array}{lcl}
P_y & \longleftarrow & P \uparrow Q \uparrow P \\
Q_y & \longleftarrow & Q \uparrow P \uparrow Q
\end{array}
$$

Note: dimensional “stretching” implied via index notation (y)

Arithmetic Formulation of the Peano Function

In addition to the classical 2D-construction in the "nonal" system, there is also a dimension-splitting approach based on ternary system:

$t = 0_3.t_1t_2t_3t_4\ldots$, then

$$p(0_3.t_1t_2t_3t_4\ldots) = P^x_{t_1} \circ P^y_{t_2} \circ P^x_{t_3} \circ P^y_{t_4} \circ \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

with the operators

$$P^x_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+0 \\ \frac{1}{3}y+0 \end{pmatrix} \quad P^x_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+1 \\ \frac{1}{3}y+\frac{1}{3} \end{pmatrix} \quad P^x_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+0 \\ \frac{1}{3}y+\frac{2}{3} \end{pmatrix}$$

and P^y analogously.

Key idea: each ternary digit defines scaling in only one dimension!

Peano's Representation of the Peano Curve

Definition: (Peano curve, original construction by G. Peano)

- each $t \in \mathcal{I} := [0,1]$ has a ternary representation

$$t = (0_3.t_1 t_2 t_3 t_4 \ldots)$$

- define the mapping $p \colon \mathcal{I} \to \mathcal{Q} := [0,1] \times [0,1]$ as

$$p(t) := \begin{pmatrix} 0_3.t_1\, k^{t_2}(t_3)\, k^{t_2+t_4}(t_5) \ldots \\ 0_3.k^{t_1}(t_2)\, k^{t_1+t_3}(t_4) \ldots \end{pmatrix}$$

  where $k(t_i) := 2 - t_i$ for $t_i = 0, 1, 2$ and k^j is the j-times concatenation of the function k

Peano's Representation of the Peano Curve (2)

Still to prove:

- p is independent of the ternary representation
- the Peano curve $p : \mathcal{I} \to \mathcal{Q}$ defines a space-filling curve.

Comments:

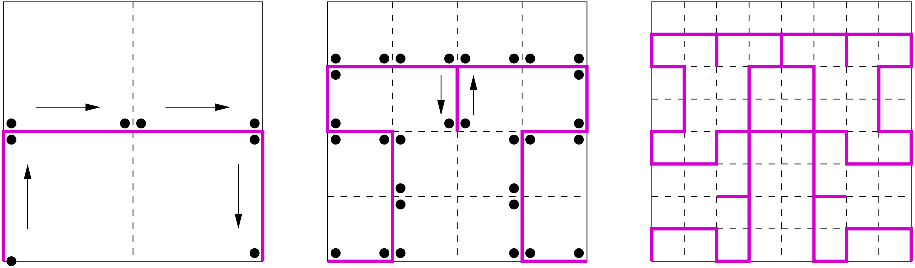
- the direction of "switchback" can be both vertical (see definition), horizontal, or mixed;
- actually, **272 different Peano curves** of the switchback type can be constructed using the same principles;
  For comparison: there are only two different 2D Hilbert curves
- in addition: Peano-Meander curves

Part III

Fractal Curves

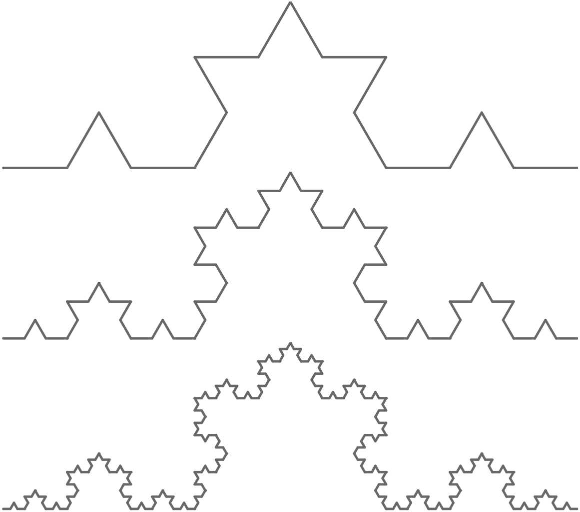
Recall: Approximating Polygons

First approximating polygons of the Hilbert curve:



- polygon results from recursive repetition of a basic pattern
  $\rightarrow$ “Leitmotiv”
- note: Leitmotiv “added” to alternating sides of the polygon (compare location of entry/exit points in the illustration)
- strong similarity to **Fractal curves**

Example: Koch Curve



How Long are Approximating Polygons?

Example: Hilbert curve

- polygon results from recursive repetition of the Leitmotiv
- every recursion step **doubles** the length of the polygon in each subsquare

  $\Rightarrow$ length of the n-th polygon is $2^n \to \infty$ for $n \to \infty$.

Corollaries:

- the “length” of the Hilbert curve is not well defined
- instead, we can give an “area” of the Hilbert curve (1, the area of the unit square)

$\Rightarrow$ **Question: what’s the dimension of a Hilbert curve?**

Fractal Dimension of Curves

Measuring the length of a curve:

- approx. the curve by a polygon with faces of length ϵ
  $\Rightarrow$ gives a measured length $L(\epsilon)$.
  (cmp. approximating polygons of a space-filling curve)
- in case of recursive repeat of a Leitmotiv:
  replace each units of length r by a polygon of length q, then

$$L\left(\tfrac{\epsilon}{r}\right) = \frac{q}{r} L(\epsilon), \qquad L(1) := \lambda$$

- we obtain for the length $L(\epsilon)$:

$$L(\epsilon) = \lambda \epsilon^{1-D}, \qquad \text{where} \quad D = \log_r q = \frac{\log q}{\log r}$$

Fractal Dimension of Curves (2)

Length of a recursively defined curve computed as

$$L(\epsilon) = \lambda \epsilon^{1-D}, \qquad \text{mit} \quad D = \log_r q = \frac{\log q}{\log r}$$

$\Rightarrow$ D is the **fractal dimension** of the curve

$\Rightarrow$ λ is the lenth w.r.t. that dimension

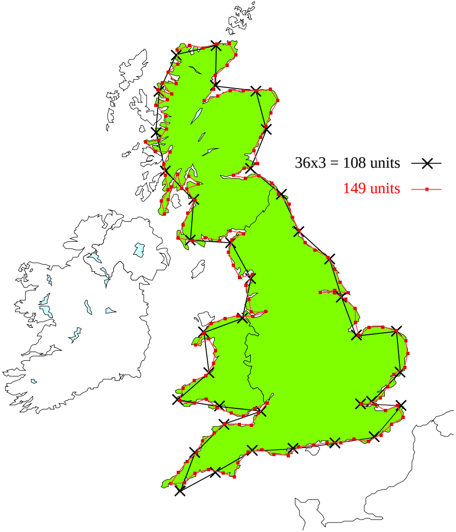
Gives “well defined” dimension:

- in all other “dimensions”, the length is 0 or ∞!
- the fractal dimension of the 2D Hilbert curve is 2, similar for the Peano curve

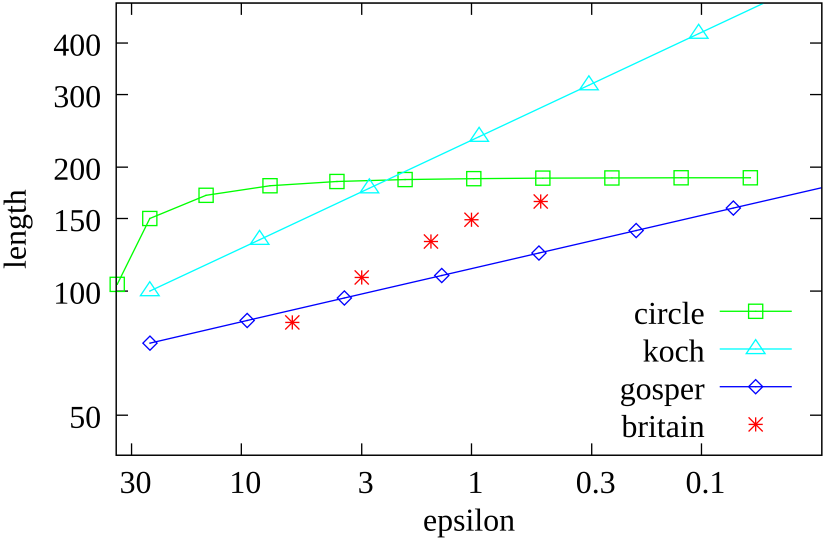
$\rightarrow$ **Hausdorff dimension**

How Long is the Coastline of Britain?

Compare, e.g., Mandelbrot: The Fractal Geometry of Nature

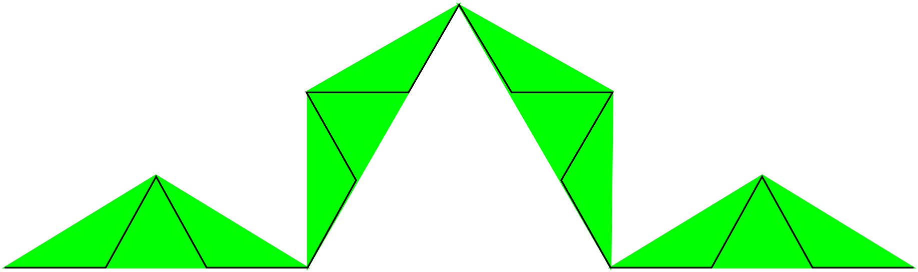


Test: Length of Fractal Curves



Exercise: What is the Area of a Fractal Curve?

Koch curve as example:



$\rightarrow$ refine green area and compute its limit value ...