

# Algorithms of Scientific Computing

## 1 Project: Interpolation of the Trajectory of the Asteroid Pallas<sup>1</sup>

Motivated by the discovery of the asteroids Ceres (1801) and Pallas (1802), Carl Friedrich Gauss studied the computation of planet trajectories in the beginning of the 19th century. There, he was faced with the following problem of trigonometric interpolation.

### Interpolation of the Asteroid's Trajectory

The following data of the trajectory have been available to Gauss:

Ascension $\theta$ (in degrees)	0	30	60	90	120	150
Declination $X$ (in minutes)	408	89	-66	10	338	807
Ascension $\theta$ (in degrees)	180	210	240	270	300	330
Declination $X$ (in minutes)	1238	1511	1583	1462	1183	804

Since the declination  $X$  is periodic with regard to  $\theta$ , the given trajectory data should be interpolated by the following trigonometric function:

$$X(\theta) = a_0 + \sum_{k=1}^5 \left( a_k \cos \left( \frac{2\pi k \theta}{360} \right) + b_k \sin \left( \frac{2\pi k \theta}{360} \right) \right) + a_6 \cos \left( \frac{2\pi \cdot 6\theta}{360} \right) \quad (1)$$

The data  $X_l$  and  $\theta_l = 30l$  have to satisfy  $X(\theta_l) = X_l$  for all  $l = 0, \dots, 11$ . Thus,

$$X_l = a_0 + \sum_{k=1}^5 \left( a_k \cos \left( \frac{\pi k l}{6} \right) + b_k \sin \left( \frac{\pi k l}{6} \right) \right) + a_6 \cos(\pi l). \quad (2)$$

### Python Demo

Create a Python script (using Jupyter Notebook) to compute the coefficients  $a_k$  and  $b_k$ . Plot the graph of the interpolated trajectory.

Note: There will be a respective introduction to Jupyter Notebook in the first tutorials.

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<sup>1</sup>Project idea and data are taken from: W. L. Briggs, Van Emden Henson, *The DFT – An Owner's Manual for the Discrete Fourier Transform*, SIAM, 1995

## Exercise 1

Show that the interpolation problem in equation (2) is equivalent to

$$X_l = \sum_{k=-5}^6 c_k e^{i2\pi kl/12}, \quad (3)$$

if  $a_k$  and  $b_k$  are chosen as  $a_k = 2\text{Re}\{c_k\}$  and  $b_k = -2\text{Im}\{c_k\}$  for  $k = 1, \dots, 5$ , while  $c_0 = a_0$  and  $c_6 = a_6$ . Use the special property that all  $X_l \in \mathbb{R}$  and therefore  $c_{-k} = c_k^*$ .

## Python Demo

Equation (3) also results from an interpolation problem with the complex interpolation function

$$C(x) = \sum_{k=-5}^6 c_k e^{ikx} \quad (4)$$

and the supporting points  $x_n = 2\pi n/N$ .

Use Python to compute and plot the interpolation function  $C(x)$ . Use the  $a_k$  and  $b_k$  from Exercise 1 and construct the  $C_k$  for all  $k = -\frac{N}{2} + 1, \dots, \frac{N}{2}$ . Can  $C(x)$  be used to describe the asteroid's trajectory?

## Small Dictionary of Astronomy

**Declination:** The angle between the celestial object and the celestial equator (projection of the Earth's equator onto the celestial sphere).

**(Right) Ascension:** The angle between the First Point of Aries (the point where the ecliptic intersects the celestial equator) and the intersection point of the meridian of a celestial object and the celestial equator. It is equivalent to the geographical longitude but is measured to the east on the celestial equator. The units are usually hours, minutes and seconds, where 24 hours are equal to  $360^\circ$ .