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Worksheet 3
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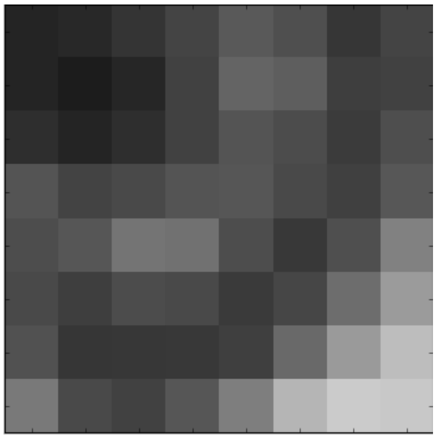
Algorithms of Scientific Computing

Discrete Cosine Transform

Exercise 1: Simple JPEG Encoder

The Discrete Cosine Transform is the key to JPEG compression. Blocks of 8×8 pixels are transformed to the frequency domain to be in a format which is more suited for compression. The code template provides you with the pixel values of the given 8×8 block of the well-known test image *Lena*.



a) In a greyscale image, the pixel values are usually encoded with 8 bit with values in $[0,255]$. The DCT, however, works on $[-128,127]$. Write a function that normalises the pixel values.

b) The DCT transforms an 8×8 block from the spatial domain to the frequency domain. Use

$$F_{uv} = \frac{1}{\sqrt{2N}} c_u c_v \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f_{xy} \cos\left(\frac{(2x+1)u\pi}{2N}\right) \cos\left(\frac{(2y+1)v\pi}{2N}\right)$$

where

$$c_{u;v} = \begin{cases} \frac{1}{\sqrt{2}} & u;v=0 \\ 1 & \text{otherwise.} \end{cases}$$

What is the complexity of your DCT routine? What is the meaning of F_{00}?

c) The coefficients F_{uv} are divided by quantisation values Q_{uv} and rounded to the nearest integer. Quantisation is a lossy process; high quantisation coefficients result in a high compression factor, though at the expense of image quality. Implement the quantisation step. A common choice for the quantisation matrix is given in the code template. As mentioned in the lecture, a full JPEG encoder would now apply further compression techniques.

d) Derive the IDCT and write a decoder for our simple JPEG block encoder. Compare the original image with the subsequently encoded and then decoded image.

Exercise 2: Discrete Cosine Transform

We start with a dataset $(f_{-N+1},\dots,f_N) \in \mathbb{R}^{2N}$, which fulfils the following symmetry constraint:

$$f_{-n} = f_n \qquad \text{for } n = 1,\dots,N-1$$

a) Show that the corresponding Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \tag{1}$$

are real values only and can be written as:

$$F_k = \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi n k}{N}\right) + \frac{1}{2} f_N \cos(\pi k) \right). \tag{2}$$

b) Show that the F_k is symmetric too.

c) Let `FFT(f,N)` be a procedure that computes the coefficients F_k efficiently (according to equation (1)) from a field `f` which consists of $2N$ values f_n. (The result is written back to field `f`)

Write a short procedure `DCT(g,N)` which uses procedure `FFT` to compute the coefficients F_k for $k = 0,\dots,N$ from equation (2) for the (non-symmetrical) data $f_0,\dots,f_N$, stored in the parameter field g.

Exercise 3: Fast Discrete Cosine Transform

Determine the butterfly scheme for equation (1) from the previous exercise. Divide the dataset f_n of length $2N$ into a dataset $g_n := f_{2n}$, containing all values with an even index, and a dataset $h_n := f_{2n-1}$, with all values with an odd index. Which symmetries can be found in g_n and h_n? Of which kind (Cosine/Sine Transformation, DFT with real data) are the according DFTs of length N? Which symmetries can be found if the dataset f_n fulfils the following symmetry constraint:

$$f_{-n} = f_{n+1}.$$