Algorithms of Scientific Computing

Exercise 1: Fourier Series

Compute the Fourier coefficients for the following three periodic functions. What is their Fourier series?

a) Repeating Dirac delta.

b) Odd square wave.

c) Repeating ramp.

Exercise 2: DFT and Least Square Approximation

For a given $N \in \mathbb{N}$ and $A \in \mathbb{R}$, let $\Delta x = A/N$.

$\forall n, 0 \leq n < N, x_n = n\Delta x$ and $f_n \in \mathbb{C}$ form the data pair $(x_n, f_n)$. Note that the $x_n$ are the equally spaced point of the interval $[0, A - \Delta x]$.

We want to find an approximation to the data using the $N$-trigonometric polynomial $\phi_N$, given by

$$\phi_N(x) = \sum_{k=0}^{N-1} \alpha_k e^{i2\pi kx/A}, \quad (1)$$

the function $\phi_N$ is called a trigonometric polynomial because it is a polynomial in the quantity $e^{i2\pi x/A}$.
The approximation should fit the least square criterion, where we minimize the discrete least squares error \( E \) defined as
\[
E = \sum_{n=0}^{N-1} |f_n - \phi_N(x_n)|^2
\]  
(2)

Find the \( N \) coefficients \( \alpha_0, \ldots, \alpha_{N-1} \). Do you know an algorithm to compute them efficiently ?

Hint: use the expression for the partial derivative of the error \( E \):
\[
\frac{\partial E}{\partial \alpha_k} = \sum_{n=0}^{N-1} e^{-i2\pi nk/N} \left( f_n - \sum_{p=0}^{N-1} \alpha_pe^{i2\pi np/N} \right),
\]  
(3)

and set these derivatives to 0.

**Exercise 3: Fast Discrete Sine Transform**

Formulate the butterfly scheme for the equation
\[
F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn},
\]  
(4)

where the dataset \( f_{-N+1}, \ldots, f_N \in \mathbb{R} \) fulfils the following symmetry constraint:
\[
f_{-n} = -f_n
\]

Split the dataset \( f_n \) of length \( 2N \) into a dataset \( g_n := f_{2n} \), containing all values with an even index, and a dataset \( h_n := f_{2n-1} \), with all values with an odd index. What symmetries can be found in \( g_n \) and \( h_n \)?