Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)

Numerical Quadrature for One-dimensional Functions

This week we focus on the question how to determine the definite integral

\[ F(f, a, b) = \int_a^b f(x) \, dx \quad \text{for functions } f : [a, b] \to \mathbb{R}. \]

It is not always possible to integrate the function \( f \) analytically, and so we use numerical quadrature. For the implementation of our algorithms we will use the Python language.

Exercise 1: Analytical Integration

Consider the functions

\[ f(x) = -4x(x-1) \quad (1) \]
\[ g(x) = \frac{8}{9} \cdot (-16x^4 + 40x^3 - 35x^2 + 11x). \quad (2) \]

Compute the antiderivatives and evaluate the integrals.

Hint: If not specified otherwise, the domain considered from now on is the unit interval \( \Omega = [0, 1] \).

Exercise 2: Composite Trapezoidal Rule

Write a function that approximates the integral via the Composite Trapezoidal Rule.

Exercise 3: Composite Simpson Rule

Do the same as in Exercise 2, only use the Composite Simpson Rule now.
Excercise 4: Archimedes’ Hierarchical Approach

Again we focus on the question how to determine the definite integral

\[ F(f, a, b) = \int_a^b f(x) \, dx \quad \text{for functions } f : [a, b] \to \mathbb{R}. \]

In this exercise we will use Archimedes’ approach to approximate the integral.

Let \( \vec{u} = [u_0, \ldots, u_n]^T \in \mathbb{R}^n, n = 2^l - 1, l \in \mathbb{N} \) a vector of function values with \( u_i = f(x_i = i + \frac{1}{2^l}). \)

\( a) \) Write a function that transforms a given vector \( \vec{u} \in \mathbb{R}^n \) to a similar vector \( \vec{v} \in \mathbb{R}^n \) containing the hierarchical coefficients needed for Archimedes’ quadrature approach. \textbf{Hint:} Later in the lecture we will officially call this process “hierarchization”, thus the function name.

\( b) \) Having computed the vector \( \vec{v} \) with the hierarchical coefficients, implement a function evaluating the integral.

\( c) \) Write a function “dehierarchize1d” similar to “hierarchize1d” that computes the inverse of the transformation above.

Excercise 5: Thoughts about Adaptivity

Discuss how the previous methods could be extended in order to improve their approximation quality.