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Worksheet 6
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Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

Hierarchical Basis and Finite Element Method

Exercise 1: Interpolation with Hierarchical Basis

Given a set of m data points

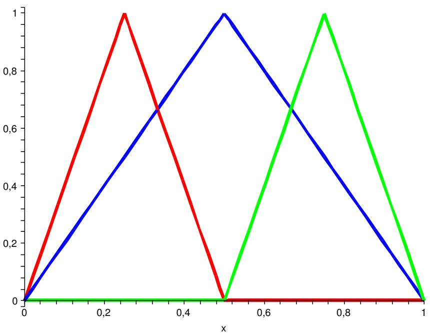
$$\{(x_1, u_1), \ldots, (x_m, u_m)\},$$

where $u_i = f(x_i), \forall i = 1, \ldots, m$.

Interpolate/approximate the underlying function $f \in \mathscr{V}$ with $f_n \in \mathscr{V}_n$ of the form

$$f(x) \approx f_n(x) = \sum_{k=1}^{n} c_k \psi_k(x)$$

where $\{\psi_k(x)\}$ are hierarchical basis as shown in figure.



(a) Construct a linear system of equations to obtain c_k for $n = 3$.

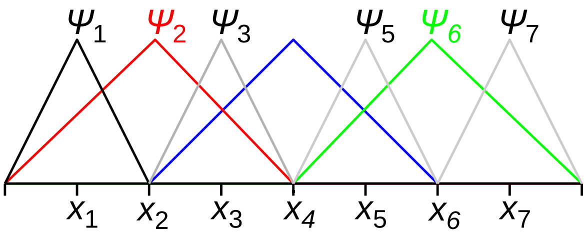
(b) Derive formulas for c_k (dependent on f_n).

(c) Repeat tasks (a) and (b) for $n = 7$.

(d) Extend the interpolation to arbitrary data poitns $n = 2^L - 1$.

Exercise 2: Interpolation with Partial-hierarchical Basis

Consider the following partial hierarchical basis:



Again, solve the interpolation problem:

(a) Construct a linear system of equations to obtain c_k for $n = 7$.

(b) Extend the system to $n = 2^L - 1$.

(c) How does this extend to solving the interpolation problem for a hierarchical basis?

Exercise 3: Function approximation

In problems of function interpolation or regression, we are given a set of data points $\{(x_1, y_1), \ldots, (x_m, y_m)\}$, we seek for a function $f \in \mathbb{V}$, such that the distance between this function and all data points are minimized, e.g.,

$$f(x) = \arg\min_f \left\{ \sum_{j=1}^{m} \|y_j - f(x_j)\| \right\}. \tag{1}$$

We can express f in the form of

$$f(x) = \sum_i \alpha_i \cdot \phi_i(x), \tag{2}$$

where $\{\phi_i\}$ is a set of known basis functions that span the function space, i.e., $\mathbb{V} = \text{span}\{\phi_i\}$. Now the problem of searching for f has become searching for $\{\alpha_i\}$, the coefficients of basis functions.

In this exercise, we would like to employ the same strategy to approximate a function $g(x)$. The goal is to look for an f of the form (2) that can best approximate $g(x)$, e.g.,

$$f(x) = \arg\min_f \|g(x) - f(x)\|_2. \tag{3}$$

1. Show that solving (3) is equivalent to finding a set of coefficients $\{\alpha_k\}$, where

$$\int_{-\infty}^{+\infty} \phi_k(x) \left(g(x) - \sum_i \alpha_i \cdot \phi_i(x) \right) \mathbf{d}x \stackrel{!}{=} 0, \ \forall k \tag{4}$$

2. Transform (4) into a linear system of the form $A\alpha = b$. What does A look like?

3. Let $g(x) = -4x^2 + 4x$, where $x \in [0, 1]$. Solve (4) with nodal basis hat functions and piecewise constant functions.