- $$f_N \stackrel{!}{=} \underset{f_N \in V_N}{\operatorname{argmin}} \Big(\underbrace{\frac{1}{m}\sum_{i=1}^{m} (y_i - f_N(x_i))^2}_{H_1(\vec{v})} + \underbrace{\lambda \|\nabla_x f_N\|_{L_2}^2}_{H_2(\vec{v})} \Big),$$

where $\boxed{f_N(x) = \sum_{j=1}^{N} v_j \varphi_j(x)}$ ← $\{v_j\}$ is unknown, so finding f_N is equi. to finding $\vec{v} \in \mathbb{R}^N$

- Let $H_1(\vec{v}) := \frac{1}{m}\sum_{i=1}^{m}(y_i - f_N(x_i))^2$ and $H_2(\vec{v}) := \lambda \|\nabla_x f_N\|_{L_2}^2$

$$\vec{v} \stackrel{!}{=} \underset{\vec{v} \in \mathbb{R}^N}{\operatorname{argmin}} \big(\underbrace{H_1(\vec{v})}_{\text{convex}} + \underbrace{H_2(\vec{v})}_{\text{convex}} \big)$$

▲ sum of convex funcs are convex

Minimize ⇒ set partial derivative w.r.t. all v_j to 0
(finding global minimum of a convex func.)

$$\frac{\partial}{\partial v_k}\big(H_1(\vec{v}) + H_2(\vec{v})\big) \stackrel{!}{=} 0, \quad \forall v_k$$

$$\Downarrow$$

$$\frac{\partial}{\partial v_k} H_1(\vec{v}) + \frac{\partial}{\partial v_k} H_2(\vec{v}) \stackrel{!}{=} 0$$

$$\Downarrow$$

see pg.2 see pg.2

$$\underbrace{\frac{2}{m}\sum_{i=1}^{m}\Big[\sum_{j=1}^{N}\varphi_j(x_i)\varphi_k(x_i)\, v_j - \varphi_k(x_i)\, y_i\Big]}_{\frac{\partial}{\partial v_k} H_1} + \underbrace{2\lambda \sum_{j=1}^{N} C_{jk} v_j}_{\frac{\partial}{\partial v_k} H_2} \stackrel{!}{=} 0, \quad C_{jk} = \int_\Omega \varphi_j'(x)\varphi_k'(x)\,dx$$

$$\times \frac{m}{2} \Big\Downarrow \text{ move all } v_j \text{ to one side}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{N}\varphi_j(x_i)\varphi_k(x_i)\cdot v_j + \lambda m \sum_{j=1}^{N} C_{jk} v_j \stackrel{!}{=} \varphi_k(x_i)\, y_i, \quad \forall v_k$$

$$\Downarrow \text{ matrix form}$$

$$G^T G v + \lambda m C v = G^T y$$

$$\underbrace{(G^T G + \lambda m C)}_{\substack{A \\ N\times N}} \underbrace{v}_{N} = \underbrace{G^T}_{N\times m} \underbrace{y}_{m}$$

- $G_{ij} = \varphi_j(x_i) \to G: m \times N$, $G^T: N \times m$
- $C_{jk} = \int_\Omega \varphi_j'(x)\varphi_k'(x)\,dx$ — $N \times N$

$$H_1(\vec{v}) = \frac{1}{m}\sum_{i=1}^{m}\left(y_i - \sum_{j=1}^{N} v_j\varphi_j(x_i)\right)^2$$

$$\frac{\partial}{\partial v_k} H_1(\vec{v})$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left[\frac{\partial}{\partial v_k}\left(y_i - \sum_{j=1}^{N} v_j\varphi_j(x_i)\right)^2\right] \longleftarrow \quad \frac{d}{dx}(a-x)^2 = 2(a-x)\cdot\frac{d(a-x)}{dx}$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left[2\left(y_i - \sum_{j=1}^{N} v_j\varphi_j(x_i)\right)\frac{\partial}{\partial v_k}\left(\cancel{y_i}^{\,0} - \cancel{v_1\varphi_1(x_i)}^{\,0} - \cdots - v_k\varphi_k(x_i) - \cdots - \cancel{v_N\varphi_N(x_i)}^{\,0}\right)\right]$$

$$= \frac{2}{m}\sum_{i=1}^{m}\left[\sum_{j=1}^{N}\varphi_j(x_i)\varphi_k(x_i)v_j - \varphi_k(x_i)y_i\right] \qquad \frac{\partial}{\partial v_k}\left(-v_k\varphi_k(x_i)\right) = -\varphi_k(x_i)$$

$$H_2(\vec{v}) = \lambda\|\nabla_x f_N\|^2_{L_2} \qquad \|f\|_{L_2} = \sqrt{\int_\Omega f^2 dx}$$

$$= \lambda\int_\Omega(\nabla_x f_N)^2 dx \qquad f_N = \sum_j v_j\varphi_j(x)$$

$$= \lambda\int_\Omega\left(\sum_{j=1}^{N} v_j\varphi_j'(x)\right)^2 dx \qquad \nabla_x f_N(x) = \sum_j v_j\varphi_j'(x)$$

$$\frac{\partial}{\partial v_k} H_2(\vec{v})$$

$$= \lambda\int_\Omega\left[\frac{\partial}{\partial v_k}\left(\sum_{j=1}^{N} v_j\varphi_j'(x)\right)^2\right]dx \qquad \frac{d}{dx}(f(g)) = \frac{df}{dg}\cdot\frac{dg}{dx}$$

$$= \lambda\int_\Omega\left[2\left(\sum_{j=1}^{N} v_j\varphi_j'(x)\right)\cdot\underbrace{\frac{\partial}{\partial v_k}\left(v_1\varphi_1'(x) + \cdots + v_k\varphi_k'(x) + \cdots v_N\varphi_N'(x)\right)}_{\varphi_k'(x)}\right]dx$$

has nothing to do with x

$$= 2\lambda\int_\Omega\left(\sum_{j=1}^{N} v_j\varphi_j'(x)\varphi_k'(x)\right)dx$$

$$= 2\lambda\sum_{j=1}^{N}\underbrace{\left(\int_\Omega\varphi_j'(x)\varphi_k'(x)\,dx\right)}_{C_{jk}}\cdot v_j$$

$$= 2\lambda\sum_{j=1}^{N} C_{jk}\cdot v_j, \quad C_{jk} = \int_\Omega\varphi_j'(x)\varphi_k'(x)\,dx = \begin{cases} a, & j=k \\ 0, & j\neq k\end{cases}$$

known