Algorithms for Scientific Computing

Wavelets

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Part I

Haar Wavelets as an Hierarchical Basis
Remember the 1D Hierarchical Basis

• “mother of all hat functions”: \( \phi(x) := \max\{1 - |x|, 0\} \)
• hat functions on level \( l \in \mathbb{N} \) with mesh width \( h_l = 2^{-l} \)
  at grid points \( x_{l,i} = i \cdot h_l \):
  \[
  \phi_{l,i}(x) := \phi \left( \frac{x - x_{l,i}}{h_l} \right)
  \]
• hierarchical basis functions on level \( l \):
  \( \phi_{l,i}(x) \) for all \( i \in I_l := \{i : 1 \leq i < 2^l, \ i \text{ odd}\} \)
• resulting hierarchical basis
  \[
  \psi_n := \bigcup_{l=1}^{n} \{\phi_{l,i} : i \in I_l\}.
  \]
• with corresponding function spaces:
  \[
  W_l := \text{span} \{\phi_{l,i} : i \in I_l\} \quad \text{and} \quad V_n = \bigoplus_{l=1}^{n} W_l
  \]
Hierarchical vs. Nodal Basis

→ for piecewise linear (basis) functions

Now: how to build a piecewise constant basis?
Piecewise Constant Basis – Attempt # 1
Discussion:

- obviously qualifies as a “hierarchical basis” w.r.t. hierarchical levels and mesh widths
- built from a “mother of all step functions”, e.g.:
  \[
  \phi(x) := \begin{cases} 
  1 & \text{if } 0 < x < 1 \\
  0 & \text{otherwise}
  \end{cases}
  \]
- hierarchical basis functions on level \( l \):
  \[
  \phi_{l,i}(x) := \phi \left( \frac{x - x_{l,i}}{h_l} \right)
  \]
- nodal basis on level \( l \):
  \[
  V_l = \text{span}\{\phi_{l,i}(x) : i = 0, \ldots, 2^l - 1\}
  \]
- hierarchical surplus:
  \[
  W_l = \text{span}\{\phi_{l,i}(x) : 1 \leq i < 2^l, \ i \text{ odd}\}
  \]
- would hierarchical surpluses be small in such a setting?
- are functions represented well by coarse-level basis functions?
Attempt #2: “Hierarchical Haar Basis”

- for each *interval*, we obtain a contribution from each *level*
- course-level representations will consist of *average values*
- each “surplus” level add corrections to averages
Hierarchical Haar Basis

• again a hierarchical basis with “mother Haar function”:

$$\psi(x) := \begin{cases} 
1 & \text{if } 0 < x < 1 \\
-1 & \text{if } 1 < x < 2 \\
0 & \text{otherwise}
\end{cases}$$

• hierarchical Haar basis functions on level $l$:

$$\psi_{l,i}(x) := \psi \left( \frac{x - x_{l,i}}{h_l} \right) \quad \text{for all } i \in \mathcal{I}_l := \{i : 0 \leq i < 2^l, \ i \text{ even}\}$$

• hierarchical surplus space for each level:

$$W_l := \text{span} \{\psi_{l,i} : i \in \mathcal{I}_l\}$$

• space of piecewise constant functions $V_n = \bigoplus_{l=0}^{n} W_l$

→ includes a step function on interval $(0, 1)$ for $l = 0$
Hierarchical Haar Basis – Coefficients

• consider a piecewise constant function \( \in V_1 \):

\[
s(x) := a\phi_{1,0}(x) + b\phi_{1,1}(x) \begin{cases} 
a & \text{if } 0 < x < \frac{1}{2} 

b & \text{if } \frac{1}{2} < x < 1 

0 & \text{otherwise}
\end{cases}
\]

• condition in interval \( 0 < x < \frac{1}{2} \):

\[
v_{0,0} \psi_{0,0}(x) + v_{1,0} \psi_{1,0}(x) = v_{0,0} + v_{1,0} = a
\]

• condition in interval \( \frac{1}{2} < x < 1 \):

\[
v_{0,0} \psi_{0,0}(x) + v_{1,0} \psi_{1,0}(x) = v_{0,0} - v_{1,0} = b
\]

• solve linear system of equations:

\[
v_{0,0} = \frac{1}{2}(a + b) \quad v_{1,0} = \frac{1}{2}(a - b)
\]
Hierarchical Haar Basis – Transformation

• represent a piecewise constant function \( s(x) \in V_l \):

\[
s(x) = \sum_{i=0}^{2^l-1} c_{l,i} \phi_{l,i}(x)
\]

• write as coarse function plus hierarchical surplus:

\[
s(x) = \sum_i c_{l,i} \phi_{l,i}(x) = \sum_i c_{l-1,i} \phi_{l-1,i}(x) + \sum_{i \in I_l} d_{l,i} \psi_{l,i}(x)
\]

\( \in V_l \) \( \in V_{l-1} \) \( \in W_l \)

• examine intervals \( x_{l,2i} < x < x_{l,2i+1} \) and \( x_{l,2i+1} < x < x_{l,2i+2} \):

\[
c_{l-1,i} + d_{l,2i} = c_{l,2i} \quad \text{and} \quad c_{l-1,i} - d_{l,2i} = c_{l,2i+1}
\]

• leads to formula for \( c_{l-1,i} \) and \( d_{l,2i} \) (note the even index \( 2i \)):

\[
c_{l-1,i} = \frac{1}{2} (c_{l,2i} + c_{l,2i+1}) \quad d_{l,2i} = \frac{1}{2} (c_{l,2i} - c_{l,2i+1})
\]
Part II

Haar Wavelets as Wavelets
Change of Notation – Scaling Function

• define *scaling function*:

\[
\phi(x) := \begin{cases} 
1 & \text{if } 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

• nodal basis functions on level \(l\):

\[
\phi_{l,k}(x) := 2^{l/2} \phi \left( \frac{x - x_{l,k}}{h_l} \right) = 2^{l/2} \phi \left( \frac{x - k \cdot 2^{-l}}{2^{-l}} \right) = 2^{l/2} \phi \left( 2^l x - k \right)
\]

(remember: \(x_{l,k} = k \cdot 2^{-l}\) and \(h_l = 2^{-l}\))

• scaling with \(2^{l/2}\) to be discussed . . .

• resulting nodal basis on level \(l\):

\[
V_l = \text{span}\{\phi_{l,k}(x): k = 0, \ldots, 2^l - 1\}
\]
Change of Notation – Wavelet Functions

• define **mother Haar wavelet**:

\[
\psi(x) := \begin{cases} 
1 & \text{if } 0 < x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} < x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

• **Haar wavelet functions** on level \( l \):

\[
\psi_{l,k}(x) := 2^{l/2} \psi \left( 2^l x - k \right) = 2^{l/2} \psi \left( \frac{x - 2^{-l}k}{2^{-l}} \right) = 2^{l/2} \psi \left( \frac{x - x_{l,k}}{h_l} \right)
\]

for \( k = 0, \ldots, 2^l - 1 \), (but no “stride two”)

• Important changes:
  • shifted numbering of levels: \( \psi(x) \) defined on \([0, 1]\)
  • thus: supports of \( \psi_{l,k}(x) \) and \( \psi_{l,k+1}(x) \) no longer overlap
  • index \( k = 0, \ldots, 2^l - 1 \) used with “stride 1”
• define **mother Haar wavelet**:

\[
\psi(x) := \begin{cases} 
1 & \text{if } 0 < x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} < x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

• **Haar wavelet functions** on level \( l \):

\[
\psi_{l,k}(x) := 2^{l/2} \psi \left( 2^l x - k \right) \quad \text{for } k = 0, \ldots, 2^l - 1.
\]

• wavelet space for each level:

\[
W_l := \text{span} \left\{ \psi_{l,k} : k = 0, \ldots, 2^l - 1 \right\}
\]

• definition of function spaces: \( V_{l+1} = V_l \oplus W_l \)
Haar Wavelet Functions
Haar Wavelets – Transformation

- represent a piecewise constant function \( s(x) \in V_l \):

\[
s(x) = \sum_{k=0}^{2^l-1} c_{l,k} \phi_{l,k}(x)
\]

- write as coarse function plus hierarchical surplus:

\[
s(x) = \sum_{k} c_{l,k} \phi_{l,k}(x) = \sum_{k} c_{l-1,k} \phi_{l-1,k}(x) + \sum_{k} d_{l-1,k} \psi_{l-1,k}(x)
\]

\( \in V_l \)

\( \in V_{l-1} \)

\( \in W_{l-1} \)

- transform \( c_{l,2k} \) to \( c_{l-1,k} \) and \( d_{l-1,k} \):

\[
c_{l-1,k} = \frac{1}{\sqrt{2}} (c_{l,2k} + c_{l,2k+1}) \quad d_{l-1,k} = \frac{1}{\sqrt{2}} (c_{l,2k} - c_{l,2k+1})
\]

- backward transform \( c_{l-1,k} \) and \( d_{l,2k} \) to \( c_{l,2k} \) and \( c_{l,2k+1} \):

\[
c_{l,2k} = \frac{1}{\sqrt{2}} (c_{l-1,k} + d_{l-1,k}) \quad c_{l,2k+1} = \frac{1}{\sqrt{2}} (c_{l-1,k} - d_{l-1,k})
\]
Haar Wavelets – Transformation (2)

- scheme for wavelet decomposition:

- scheme for assembly:

- Note: computational effort for transformations is only $O(N)$
### Haar Wavelets – Transformation (3)

Scheme for data structures:

<table>
<thead>
<tr>
<th></th>
<th>$c^{(J)}$</th>
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</thead>
<tbody>
<tr>
<td>$c^{(J)}$</td>
<td></td>
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</tr>
<tr>
<td>$c^{(J-1)}$</td>
<td>$d^{(J-1)}$</td>
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</tr>
<tr>
<td>$c^{(J-2)}$</td>
<td>$d^{(J-2)}$</td>
<td>$d^{(J-1)}$</td>
</tr>
<tr>
<td>$c^{(J-3)}$</td>
<td>$d^{(J-3)}$</td>
<td>$d^{(J-2)}$</td>
</tr>
</tbody>
</table>
Haar Wavelets – Orthogonality

- Haar wavelets are **orthogonal** functions:
  \[
  \int \psi_{l,i}(x) \psi_{m,j}(x) \, dx := \begin{cases} 
  1 & \text{if } l = m \text{ and } i = j \\
  0 & \text{otherwise}
  \end{cases}
  \]

- two different wavelet functions \( \psi_{l,i} \neq \psi_{l,j} \) on the same level \( l \)
  \[
  \int \psi_{l,i}(x) \psi_{l,j}(x) \, dx = 0 \quad \text{(no overlap of functions!)}
  \]

- two wavelet functions \( \psi_{l,i} \neq \psi_{m,j} \) on different levels \( l < m \)
  \[
  \int \psi_{l,i}(x) \psi_{m,j}(x) \, dx = \psi_{l,i}(x_{m,j}^+) \int \psi_{m,j}(x) \, dx = 0
  \]

- scalar product of a wavelet functions \( \psi_{l,i} \) with itself
  \[
  \int (\psi_{l,i}(x))^2 \, dx = \int_{x_{l,i}}^{x_{l,i}+2^{-l}} (2^{l/2})^2 \, dx = 1
  \]
Haar Wavelets – Summary and Next Steps

Haar wavelets:
• hierarchical basis of piecewise constant and …
• … orthogonal basis functions
• $O(N)$ effort for hierarchical transformation (compare tutorial)

Next steps:
• applications in signal and image processing
• extension to 2D (and higher dimensions)
• is there a piecewise linear/polynomial/higher-order orthogonal(!) wavelet basis?
Part III

Wavelets in Signal and Image Processing
Scaling Functions and Wavelet Functions in 2D

Use tensor product, as for hierarchical basis:

- 2D scaling functions on levels $l_1, l_2$:
  \[
  \phi_{l_1,l_2,k_1,k_2}(x_1, x_2) := \phi_{l_1,k_1}(x_1) \cdot \phi_{l_2,k_2}(x_2)
  \]

- 2D wavelet functions on levels $l_1, l_2$:
  \[
  \psi_{l_1,l_2,k_1,k_2}(x_1, x_2) := \psi_{l_1,k_1}(x_1) \cdot \psi_{l_2,k_2}(x_2)
  \]

- thus straightforward extension to 3D and higher dimensions
- 2D transform equivalent to sequence of 1D transforms
  (same as for Fourier Transforms and also for Hierarchical Basis)
2D Wavelets – A Single Transformation Step

\[ \begin{array}{c}
cc2 \\
\downarrow \\
cc1 \\
dc2 \\
dd2 \\
\end{array} \rightarrow \begin{array}{cc}
cc1 & cd2 \\
\downarrow & \downarrow \\
dc2 & dd2 \\
\end{array} \]
2D Wavelets – Storage Scheme

<table>
<thead>
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<th>cd1</th>
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<tbody>
<tr>
<td>dc1</td>
<td>dd1</td>
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<table>
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<tr>
<th>cd2</th>
<th>cd3</th>
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<tbody>
<tr>
<td>dc2</td>
<td>dd2</td>
</tr>
</tbody>
</table>

| dc3 | dd3 |
Wavelet-Based Compression of Image Data

Typical steps for image compression:

1. Conversion of colour model
   (separation of brightness and colour information)
2. 2D discrete Wavelet transform
3. Quantisation of the coefficients (→ reduce information)
4. efficient encoding
   (loss-less compression of the quantised coefficients)

In practice:

- different algorithms: EZF, SPIHT, …
- similar to JPEG, but often much better quality
- see, e.g., Walker: “Wavelet-based Image Compression” for full details
Example: 3D Image Compression

(wavelet-based compression of raster data, A. Dehmel)
Example: 3D Image Compression (2)

(wavelet-based compression of raster data, A. Dehmel)
From Fourier Transform to Wavelets

(Discrete) Fourier Transform:

\[ f(x) \sim \sum c_k e^{i k x} \quad \text{or} \quad f_n = \sum F_k e^{i \pi k n / N} \]

- \( f \) contains only spatial information
- \( c_k, F_k \) contain only frequency information
- no relation between frequency and location

Windowed Fourier Transform:

\[ f(x) = \frac{1}{2\pi} \int \int F(u, k) g(x - u) e^{i k x} \, dk \, du, \quad F(u, k) = \int f(x) g(x - u) e^{-i k x} \, dx \]

- \( F(u, k) \): frequency \( k \) at location \( u \)
- \( g(\xi) \) a window function
  → narrow windows do not allow to locate coarse frequencies
  → but wide windows decrease accuracy in location
From Fourier Transform to Wavelets (2)

Continuous Wavelet Transform:

\[ W(a, b) = \int f(t) \psi_a^b(t) \, dt \quad \text{and} \quad f(x) = \frac{1}{C_\psi} \int \int W(a, b) \frac{\psi_a^b(x)}{a^2} \, da \, db \]

- continuous in \(a\) and \(b\)
- \(\psi_a^b(t) = |a|^{-1/2} \psi \left( \frac{t-b}{a} \right)\) with “mother wavelet” \(\psi\)
- infinitely many (redundant) coefficients \(\rightarrow\) computationally not feasible

Multiresolution Analysis/Discrete Wavelet Transform:

- restrict \((a, b)\) to discrete values \((a, b) := \left( \frac{1}{2^j}, \frac{k}{2^j} \right)\)
- thus discrete wavelet functions:

\[ \psi_{j,k} = \psi_{k}^{2^{-j}} = 2^{j/2} \psi(2^j t - k) \]

- combines frequency and location: higher spatial resolution for higher frequencies
Part IV

More Complicated Wavelets

Reference/more details ~ Aboufadel & Schlicker: Discovering Wavelets
Mother and Father Wavelets – General Situation

- **mother wavelet** $\psi(x)$
- **father wavelet** $\phi(x)$, also called **scaling function**
- basis built from scaling functions on each level $l$:

$$\phi_{l,k}(x) := 2^{l/2} \phi \left( 2^l x - k \right) \quad V_l := \text{span} \{ \phi_{l,k}(x) \}$$

- surplus basis built from wavelet functions on each level $l$:

$$\psi_{l,k}(x) := 2^{l/2} \psi \left( 2^l x - k \right) \quad W_l := \text{span} \{ \psi_{l,k}(x) \}$$

- definition of function spaces: $V_{l+1} = V_l \oplus W_l$
- wavelet basis functions are **orthonormal**:

$$\langle \psi_{l,k}(x), \psi_{m,j}(x) \rangle = \int \psi_{l,k}(x) \psi_{m,j}(x) \, dx = \begin{cases} 1 & \text{if } l = m \text{ and } k = j \\ 0 & \text{otherwise} \end{cases}$$

- also: scaling basis functions are orthonormal on each level
Scaling and Wavelet Functions

• note: $\phi_{l-1,k} \in V_l \supset V_{l-1}$, and also $\psi_{l-1,k} \in V_l = V_{l-1} \oplus W_{l-1}$
• hence, all $\phi_{l-1,k}$ and $\psi_{l-1,k}$ can be uniquely represented via the basis functions of $V_l$, i.e., the $\phi_{l,k}$:

$$\phi_{l-1,0}(x) = \sum_i p_i \phi_{l,i}(x) = 2^{l/2} \sum_i p_i \phi \left( 2^l x - i \right)$$

$$\psi_{l-1,0}(x) = \sum_i q_i \phi_{l,i}(x) = 2^{l/2} \sum_i q_i \phi \left( 2^l x - i \right)$$

• for efficiency: $p_i$ and $q_i$ should be non-zero for only a few $i$
• for Haar wavelets:

$$p_0 = \frac{1}{\sqrt{2}}, \quad p_1 = \frac{1}{\sqrt{2}}, \quad \text{all other } p_i = 0$$

$$q_0 = \frac{1}{\sqrt{2}}, \quad q_1 = -\frac{1}{\sqrt{2}}, \quad \text{all other } q_i = 0$$
Scaling and Wavelet Functions (2)

• do for all scaling functions $\phi_{l-1,k}$:

$$
\phi_{l-1,k}(x) = 2^{l/2} \sum_i p_i \phi \left( 2^l x - 2k - i \right)
$$

$$
2k+i \rightarrow i \quad 2^{l/2} \sum_i p_{i-2k} \phi \left( 2^l x - i \right) = \sum_i p_{i-2k} \phi_{l,i}(x)
$$

Note: $\phi_{l,k}(x) = 2^{l/2} \phi \left( 2^l x - k \right) = 2^{l/2} \phi \left( 2^l x - 2^l \frac{k}{2^l} \right) = 2^{l/2} \phi \left( 2^l \left( x - \frac{k}{2^l} \right) \right) = \phi_{l,0} \left( x - \frac{k}{2^l} \right)$
and thus: $\phi_{l-1,k}(x) = \phi_{l-1,0} \left( x - \frac{k}{2^{(l-1)}} \right) = 2^{l/2} \sum_i q_i \phi \left( 2^l \left( x - \frac{k}{2^{(l-1)}} \right) - i \right) = \ldots$

• and similar for wavelet functions: $\psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)$

• for Haar wavelets:

$p_{i-2k}$ and $q_{i-2k}$ are non-zero only for $i = 2k$ and $i = 2k + 1$:

$$
\phi_{l-1,k}(x) = \frac{1}{\sqrt{2}} \phi_{l,2k}(x) + \frac{1}{\sqrt{2}} \phi_{l,2k+1}(x)
$$

$$
\psi_{l-1,k}(x) = \frac{1}{\sqrt{2}} \phi_{l,2k}(x) - \frac{1}{\sqrt{2}} \phi_{l,2k+1}(x)
$$
Wavelet Transformations and Filtering

• consider a signal function represented on (fine) level $l+1$:

$$f_{l+1}(x) = \sum_i c_i^{(l+1)} \phi_{l+1,k}(x)$$

• and a decomposition $f_{l+1} = f_l + g_l$, where $f_l \in V_l$ and $g_l \in W_l$:

$$f_{l+1}(x) = \sum_i c_i^{(l+1)} \phi_{l+1,i}(x) = \sum_j c_j^{(l)} \phi_{l,j}(x) + \sum_j d_j^{(l)} \psi_{l,j}(x)$$

$$= \sum_j \left( c_j^{(l)} \sum_i p_{i-2j} \phi_{l+1,i}(x) \right) + \sum_j \left( d_j^{(l)} \sum_i q_{i-2j} \phi_{l+1,i}(x) \right)$$

$$= \sum_i \phi_{l+1,i}(x) \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$$

• two different representations of $f_{l+1}(x)$, but $\{ \phi_{l+1,k}(x) \}$ a basis:

$$\Rightarrow c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$$
Wavelet Transformations and Filtering (2)

• $p_i$ and $q_i$ determine transformation of coefficients:

$$c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$$

• solves assembly:
  for given $f_l$ and $g_l$ (i.e., given coefficients $c_j^{(l)}$ and $d_j^{(l)}$),
  find coefficients $c_i^{(l+1)}$ for $f_{l+1}$

• for Haar wavelets:

  even $i$:  \[ c_i^{(l+1)} = \frac{1}{\sqrt{2}} c_{i/2}^{(l)} + \frac{1}{\sqrt{2}} d_{i/2}^{(l)} \]

  odd $i$:  \[ c_i^{(l+1)} = \frac{1}{\sqrt{2}} c_{(i-1)/2}^{(l)} - \frac{1}{\sqrt{2}} d_{(i-1)/2}^{(l)} \]
Wavelet Transformations and Filtering (3)

- now: fine-level representation given as
  \[ f_{l+1}(x) = \sum_i c_{i}^{(l+1)} \phi_{l+1,i}(x) \]

- wanted: decomposition \( f_{l+1} = f_l + g_l \) with
  \[ f_l(x) + g_l(x) = \sum_j c_j^{(l)} \phi_{l,j}(x) + \sum_j d_j^{(l)} \psi_{l,j}(x) \]

- use that \( \{\phi_{l,k}(x)\} \) and \( \{\psi_{l,k}(x)\} \) are orthonormal basis for \( V_l \) and \( W_l \), and \( V_l \perp W_l \):
  \[ \Rightarrow \quad c_j^{(l)} = \langle f_{l+1}(x), \phi_{l,j}(x) \rangle = \left\langle \sum_i c_i^{(l+1)} \phi_{l+1,i}(x), \phi_{l,j}(x) \right\rangle \]
  \[ = \sum_i c_i^{(l+1)} \langle \phi_{l+1,i}(x), \phi_{l,j}(x) \rangle \]
  \[ = \ldots \]
Wavelet Transformations and Filtering (4)

• continued:

\[ c_{j}^{(l)} = \langle f_{l+1}(x), \phi_{l,j}(x) \rangle = \cdots = \sum_{i} c_{i}^{(l+1)} \langle \phi_{l+1,i}(x), \phi_{l,j}(x) \rangle \]

\[ = \sum_{i} c_{i}^{(l+1)} \left\langle \phi_{l+1,i}(x), \sum_{k} p_{k-2j} \phi_{l+1,k}(x) \right\rangle \]

\[ = \sum_{i} c_{i}^{(l+1)} \sum_{k} p_{k-2j} \left\langle \phi_{l+1,i}(x), \phi_{l+1,k}(x) \right\rangle = \sum_{i} c_{i}^{(l+1)} p_{i-2j} \]

• similar computation for \( d_{j}^{(l)} \), and therefore:

\[ c_{j}^{(l)} = \sum_{i} p_{i-2j} c_{i}^{(l+1)} \quad d_{j}^{(l)} = \sum_{i} q_{i-2j} c_{i}^{(l+1)} \]

• again, for Haar wavelets:

\[ c_{j}^{(l)} = \frac{1}{\sqrt{2}} c_{2j}^{(l+1)} + \frac{1}{\sqrt{2}} c_{2j+1}^{(l+1)} \quad d_{j}^{(l)} = \frac{1}{\sqrt{2}} c_{2j}^{(l+1)} - \frac{1}{\sqrt{2}} c_{2j+1}^{(l+1)} \]
Wavelet Transformations and Filtering – Summary

Wanted: decomposition $f_{i+1} = f_i + g_i$ with

- coarser representation $f_i(x) = \sum c_j^{(l)} \phi_{l,j}(x)$ with
  \[ c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)} \]

  corresponds to a **low-pass filter** (averaging)

- oscillatory surplus $g_i(x) = \sum d_j^{(l)} \psi_{l,j}(x)$ with
  \[ d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)} \]

  corresponds to a **high-pass filter** (difference computation)

- and reconstruction: $c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right)$
How to Determine the Filtering Coefficients?

- we need coefficients for low-pass and high-pass filter:

\[ c_j^{(l)} = \sum_i p_{i-2j} c_i^{(l+1)} \quad d_j^{(l)} = \sum_i q_{i-2j} c_i^{(l+1)} \]

- reconstruction then: \( c_i^{(l+1)} = \sum_j \left( p_{i-2j} c_j^{(l)} + q_{i-2j} d_j^{(l)} \right) \)

- requires **scaling equation** for scaling and wavelet functions:

\[
\phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x) \quad \psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)
\]

- requires **orthogonal** scaling and wavelet functions:
  - \( \phi_{l,k} \perp \phi_{l,j} \) and \( \psi_{l,k} \perp \psi_{l,j} \) for \( k \neq j \)
  - \( \psi_{l,k} \perp \phi_{m,j} \) if \( m \leq l \) and arbitrary \( k, j \) (i.e., \( W_l \perp V_m \))
How to Determine the Wavelet Functions? (2)

- **scaling equation** for mother and father wavelet:
  \[
  \phi(x) = \sqrt{2} \sum_k p_k \phi(2x - k) \quad \psi(x) = \sqrt{2} \sum_k q \phi(2x - k)
  \]
  also called **dilation equation**

- for Haar wavelet:
  \[
  \phi(x) = \phi(2x) + \phi(2x - 1) \quad \psi(x) = \phi(2x) - \phi(2x - 1)
  \]

- for more complicated wavelets:
  - more than 2 non-zeros \( p_k \) (and \( q_k \))
  - \( p_k \) and \( q_k \) determined to satisfy orthogonality
  - often no analytical expression for \( \phi(x) \) and \( \psi(x) \) available
  - obtain \( \phi(x) \) and \( \psi(x) \) as solutions of the scaling equation
    → see worksheet “cranking the machine”
Towards More Complicated Wavelets

“Wish List:”

- orthonormal basis of scaling functions on each level:
  \[ \langle \phi_{l,k}(x), \phi_{l,j}(x) \rangle = \begin{cases} 1 & \text{if and } k = j \\ 0 & \text{otherwise} \end{cases} \]

- scaling/wavelet functions obey top scaling equation:
  \[
  \phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x) \quad \psi_{l-1,k}(x) = \sum_i q_{i-2k} \phi_{l,i}(x)
  \]

- scaling/wavelet functions have compact support
  \[ p_i \neq 0 \text{ only for few } i \text{ (same for } q_i) \]

- “vanishing moments” of wavelet functions:
  \[ \int \psi(t) \, dt = 0 \quad \int t\psi(t) \, dt = 0 \quad \text{etc.} \]
Towards More Complicated Wavelets (2)

orthonormal basis of scaling functions:

- on each level:
  \[
  \langle \phi_{l,k}(x), \phi_{l,j}(x) \rangle = \begin{cases} 
  1 & \text{if and } k = j \\
  0 & \text{otherwise}
  \end{cases}
  \]

- combine with scaling equation and compact support:
  \[
  \phi_{l-1,k}(x) = \sum_i p_{i-2k} \phi_{l,i}(x)
  \]
  where \( p_i \neq 0 \) only for few \( i \)

and obtain:

\[
\langle \phi_{l-1,k}(x), \phi_{l-1,m}(x) \rangle = \left\langle \sum_i p_{i-2k} \phi_{l,i}(x), \sum_j p_{j-2m} \phi_{l,j}(x) \right\rangle
\]

\[
= \sum_i p_{i-2k} \sum_j p_{j-2m} \left\langle \phi_{l,i}(x), \phi_{l,j}(x) \right\rangle = \sum_i p_{i-2k} p_{i-2m}
\]

- in particular (for \( k = m \)):
  \[
  \sum_i (p_{i-2k})^2 = \sum_i p_i^2 = 1
  \]
Towards More Complicated Wavelets (3)

• in addition – for \( k = 0 \) and arbitrary \( m \neq 0 \):
  \[
  \langle \phi_{l-1,0}(x), \phi_{l-1,m}(x) \rangle = \sum_i p_i p_{i-2m} = 0
  \]

• similar argument: scaling and wavelet functions are orthogonal!
  \[
  \langle \phi_{l-1,0}(x), \psi_{l-1,0}(x) \rangle = \left\langle \sum_i p_i \phi_{l,i}(x), \sum_j q_j \phi_{l,j}(x) \right\rangle
  = \sum_i p_i \sum_j q_j \langle \phi_{l,i}(x), \phi_{l,j}(x) \rangle = \sum_i p_i q_i \neq 0
  \]

• and wavelet functions of one level are orthogonal:
  \[
  \langle \psi_{l,k}(x), \psi_{l,m}(x) \rangle = 0 \quad \Rightarrow \sum_i q_i q_{i-2(k-m)} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}
  \]

• to satisfy these requirements: \( q_k = (-1)^k p_{1-k} \)
Daubechies Wavelets (D4)

- setting: $\phi(x) = 0$ outside of interval $[0, 3]$
  $\rightarrow$ non-zero coefficients are $p_0$, $p_1$, $p_2$, and $p_3$
- orthogonality requires $\sum p_i^2 = 1$ and $\sum p_i p_{i-2m} = 0$:
  $$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1 \quad \text{and} \quad p_0 p_2 + p_1 p_3 = 0$$
- plus vanishing moments $\int \psi(t) \, dt = 0$ and $\int t \psi(t) \, dt = 0$
  together with $q_k = (-1)^k p_{1-k}$ leads to
  $$-p_0 + p_1 - p_2 + p_3 = 0 \quad \text{and} \quad -p_1 + 2p_2 - 3p_3 = 0$$
- one solution to this system:
  $$p_0 = \frac{1 + \sqrt{3}}{4\sqrt{2}}, \quad p_1 = \frac{3 + \sqrt{3}}{4\sqrt{2}}, \quad p_2 = \frac{3 - \sqrt{3}}{4\sqrt{2}}, \quad p_3 = \frac{1 - \sqrt{3}}{4\sqrt{2}}$$
Daubechies Wavelets (D4) – Scaling Function

no analytical expression available → iterative approximation

see tutorials: → “cranking the machine”
Daubechies Wavelets (D4) – Wavelet Function
Finally: Multiresolution Analysis

Definition: Multiresolution Analysis

- nested sequence of function spaces:
  \[ \cdots \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \]

- with a scaling function \( \phi \)
  such that \( \phi(2^l x - k) \) is an orthonormal Basis of \( V_l \)
  (and \( V_l = \text{span}\{\phi_{l,k} : k \in \mathbb{Z}\} \))

- \( \bigcup V_i \) is dense in \( L^2(\mathbb{R}) \)

- \( V_i \) satisfy separation property: \( \bigcap V_i = \{0\} \)

- \( f(t) \in V_i \) if and only if \( f(2^{-l}t) \in V_0 \)

Last but not least: find coefficients \( c_k \) such that \( s(x) \approx \sum c_k \phi_{l,k}(x) \)?

→ use orthogonality: \( c_k = \langle s(x), \phi_{l,k}(x) \rangle \)
  (orthogonal projection to space \( V_l \))