

Algorithms of Scientific Computing

1 Project: Interpolation of the Trajectory of the Asteroid Pallas – Sample Solution

Python Demo

see Jupyter Notebook solution `Worksheet_1.ipynb`.

Exercise 1

$$\begin{aligned} X_l &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{i2\pi kl/N} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \\ &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) + i \Im\{c_k\} \cdot \left(\cos\left(\frac{2\pi kl}{N}\right) + i \sin\left(\frac{2\pi kl}{N}\right) \right) \end{aligned}$$

Now we sort for real and imaginary part of X_l :

$$\begin{aligned} \Re\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \\ \Im\{X_l\} &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Im\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) \end{aligned}$$

Since X_l is real $\Im\{X_l\}$ must be zero. Thus, the remaining part is

$$X_l = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

By reducing the sum to the range $k = 1, \dots, \frac{N}{2} - 1$, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} \Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) + \Re\{c_{-k}\} \cdot \cos\left(\frac{2\pi(-k)l}{N}\right) \\ - \Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right) - \Im\{c_{-k}\} \cdot \sin\left(\frac{2\pi(-k)l}{N}\right)$$

Using the symmetry of the sine and cosine functions, this can be derived to

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} (\Re\{c_k\} + \Re\{c_{-k}\}) \cdot \cos\left(\frac{2\pi kl}{N}\right) \\ + (\Im\{c_{-k}\} - \Im\{c_k\}) \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

Since $c_{-k} = c_k^*$ it is $\Re\{c_{-k}\} = \Re\{c_k\}$ and $\Im\{c_{-k}\} = -\Im\{c_k\}$. So, we get

$$X_l = \Re\{c_0\} + \Re\{c_{N/2}\} \cdot \cos(\pi l) + \sum_{k=1}^{\frac{N}{2}-1} 2\Re\{c_k\} \cdot \cos\left(\frac{2\pi kl}{N}\right) - 2\Im\{c_k\} \cdot \sin\left(\frac{2\pi kl}{N}\right)$$

For $N = 12$, $a_k = 2\Re\{c_k\}$, $b_k = -2\Im\{c_k\}$ for all $k = 1, \dots, \frac{N}{2}$, $a_0 = c_0$ and $a_{\frac{N}{2}} = c_{\frac{N}{2}}$ we get equation (2):

$$X_l = a_0 + \sum_{k=1}^5 \left(a_k \cos\left(\frac{\pi kl}{6}\right) + b_k \sin\left(\frac{\pi kl}{6}\right) \right) + a_6 \cos(\pi l)$$

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