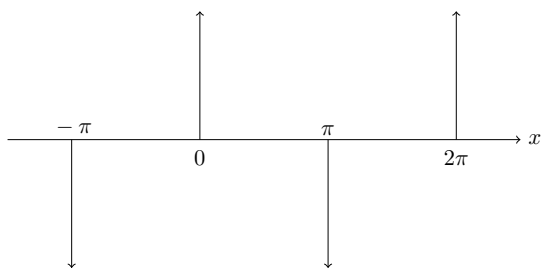


Algorithms of Scientific Computing

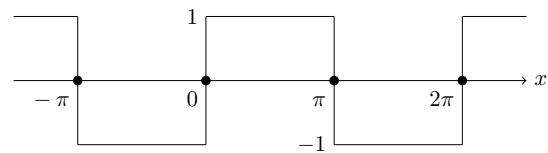
Discrete Cosine Transform

Exercise 1: Fourier Series

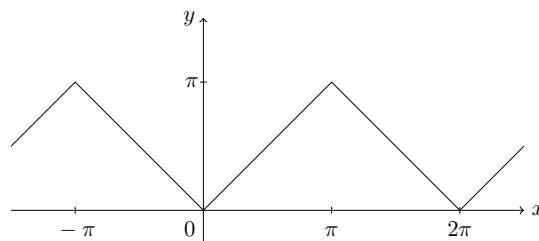
Compute the Fourier coefficients for the following three periodic functions. What is their Fourier series?



a) Repeating Dirac delta.



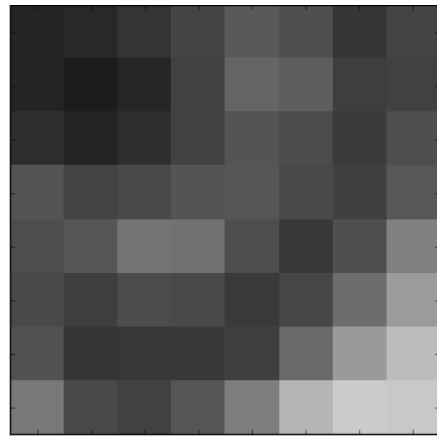
b) Odd square wave.



c) Repeating ramp.

Exercise 2: Simple JPEG Encoder

The Discrete Cosine Transform is the key to JPEG compression. Blocks of 8×8 pixels are transformed to the frequency domain to be in a format which is more suited for compression. The code template provides you with the pixel values of the given 8×8 block of the well-known test image *Lena*.



- a) In a greyscale image, the pixel values are usually encoded with 8 bit with values in $[0, 255]$. The DCT, however, works on $[-128, 127]$. Write a function that normalises the pixel values.
- b) The DCT transforms an 8×8 block from the spatial domain to the frequency domain. Use

$$F_{uv} = \frac{1}{\sqrt{2N}} c_u c_v \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f_{xy} \cos\left(\frac{(2x+1)u\pi}{2N}\right) \cos\left(\frac{(2y+1)v\pi}{2N}\right)$$

where

$$c_{u,v} = \begin{cases} \frac{1}{\sqrt{2}} & u;v = 0 \\ 1 & \text{otherwise.} \end{cases}$$

What is the complexity of your DCT routine? What is the meaning of F_{00} ?

- c) The coefficients F_{uv} are divided by quantisation values Q_{uv} and rounded to the nearest integer. Quantisation is a lossy process; high quantisation coefficients result in a high compression factor, though at the expense of image quality. Implement the quantisation step. A common choice for the quantisation matrix is given in the code template. As mentioned in the lecture, a full JPEG encoder would now apply further compression techniques.
- d) Derive the IDCT and write a decoder for our simple JPEG block encoder. Compare the original image with the subsequently encoded and then decoded image.

Exercise 3: Discrete Cosine Transform

We start with a dataset $(f_{-N+1}, \dots, f_N) \in \mathbb{R}^{2N}$, which fulfils the following symmetry constraint:

$$f_{-n} = f_n \quad \text{for } n = 1, \dots, N-1$$

- a) Show that the corresponding Fourier coefficients

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^N f_n \omega_{2N}^{-kn} \quad (1)$$

are real values only and can be written as:

$$F_k = \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi nk}{N}\right) + \frac{1}{2} f_N \cos(\pi k) \right). \quad (2)$$

b) Show that the F_k is symmetric too.

c) Let $\text{FFT}(\mathbf{f}, N)$ be a procedure that computes the coefficients F_k efficiently (according to equation (1)) from a field \mathbf{f} which consists of $2N$ values f_n . (The result is written back to field \mathbf{f})

Write a short procedure $\text{DCT}(\mathbf{g}, N)$ which uses procedure FFT to compute the coefficients F_k for $k = 0, \dots, N$ from equation (2) for the (non-symmetrical) data f_0, \dots, f_N , stored in the parameter field \mathbf{g} .