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Worksheet 3
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Algorithms of Scientific Computing

Discrete Cosine Transform – Solution

Exercise 1: Fourier Series

The Fourier coefficients are given as

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, \mathrm{d}x.$$

We plug in our three functions and compute their Fourier coefficients.

a) Recall $\int f(x)\delta_{x_0}(x)\,\mathrm{d}x = f(x_0)$. The integral gives the amplitude of $f(x)$ at the delta spike. We can also shift the integral used to computed the Fourier coefficients by any given a, here chosen between $-\pi$ and 0 ($-\pi < a < 0$). We need to do this in this case because you cannot have a Dirac delta spiking at the border of an integral (it isn't defined in that case). Then the repeating Dirac delta $DD(x)$ yields:

$$\begin{aligned}
c_k &= \frac{1}{2\pi} \int_a^{a+2\pi} DD(x) e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi} \left(\int_a^{a+\pi} \delta_0(x) e^{-ikx}\,\mathrm{d}x + \int_{a+\pi}^{a+2\pi} -\delta_\pi(x) e^{-ikx}\,\mathrm{d}x \right) \\
&= \frac{1}{2\pi} \left(e^{-ik\cdot 0} - e^{-ik\cdot\pi} \right) \\
&= \frac{1}{2\pi} \left(1 - (-1)^k \right)
\end{aligned}$$

If k is even, then the coefficients are 0, otherwise the Fourier coefficients are constant with $\frac{1}{\pi}$. In other words, all odd frequencies are equally important and necessary to approximate a delta spike. The Fourier series then is

$$F(x) = \frac{1}{\pi} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} e^{ikx}.$$

We start with the complex Fourier series and apply a sequence of simplifications.

① For even k, the term $1 - (-1)^k$ vanishes. For odd k, $1 - (-1)^k$ simplifies to 2.

② Combine positive and negative terms.

③ $\cos(kx) = \frac{e^{ikx}+e^{-ikx}}{2}$

$$\begin{aligned}
F(x) &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\left(1-(-1)^k\right)e^{ikx} \\
&\overset{①}{=} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{\pi}e^{ikx} \\
&\overset{②}{=} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{\pi}e^{ikx} + \frac{1}{\pi}e^{-ikx}\right) \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{\pi}\left(e^{-ikx}+e^{ikx}\right)\frac{2}{2} \\
&\overset{③}{=} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi}\cos(kx) \\
&= \frac{2}{\pi}\sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \cos(kx)
\end{aligned}$$

b) We compute the Fourier coefficients of the square wave $SW(x)$:

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{2\pi} SW(x)e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} SW(x)e^{-ikx}\,\mathrm{d}x + \int_{\pi}^{2\pi} SW(x)e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} 1\cdot e^{-ikx}\,\mathrm{d}x + \int_{\pi}^{2\pi} -1\cdot e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\left[\frac{1}{-ik}e^{-ikx}\right]_0^{\pi} - \left[\frac{1}{-ik}e^{-ikx}\right]_{\pi}^{2\pi}\right) \\
&= \frac{1}{2\pi}\cdot 2\left(\frac{1}{-ik}e^{-ik\pi} - \frac{1}{-ik}\right) \\
&= \frac{-1}{-ik\pi}\left(1-(-1)^k\right) \\
&= \frac{-i}{k\pi}\left(1-(-1)^k\right)
\end{aligned}$$

Note that for $k = 0$, the formula wouldn't work (we get $\frac{0}{0}$), the term c_0 is actually $\int_0^{2\pi} SW(x)\,\mathrm{d}x$ which is the average value of the function over one periode, so here $c_0 = 0$

Like with the diracs, we start with the complex Fourier series and apply a sequence of simplifications.

① For even k, the term $1-(-1)^k$ vanishes. For odd k, $1-(-1)^k$ simplifies to 2.

② Combine positive and negative terms.

③ $\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$

$$
\begin{aligned}
F(x) &= \sum_{k=-\infty}^{\infty} \frac{-i}{k\pi} \left(1 - (-1)^k\right) e^{ikx} \\
&\overset{①}{=} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{-2i}{k\pi} e^{ikx} \\
&\overset{②}{=} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(\frac{-2i}{k\pi} e^{ikx} + \frac{-2i}{-k\pi} e^{-ikx} \right) \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2i}{k\pi} \left(e^{ikx} - e^{-ikx} \right) \frac{2i}{2i} \\
&\overset{③}{=} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2i \cdot 2i}{k\pi} \sin(kx) \\
&= \frac{4}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{\sin(kx)}{k}
\end{aligned}
$$

All imaginary parts vanish and we obtain the common form of the Fourier series of the square wave. As the square wave is an odd function, it is no surprise that we are left with only sines. Note that we are left with the sines because the factor $\frac{1}{k}$ gave us an opposite sign on the negative coefficient. Also note the factor $1/k$. Although the series in infinite and we need all frequencies to represent the jumps, higher coefficients are less important than lower frequencies. This is the key difference to the delta spikes.

c) Fourier coefficients of the repeating ramp $RR(x)$:

We will use the integration by part formula

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v$$

with $u(x) = x$ and $v'(x) = e^{-ikx}$

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{2\pi} RR(x)e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} RR(x)e^{-ikx}\,\mathrm{d}x + \int_{\pi}^{2\pi} RR(x)e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} xe^{-ikx}\,\mathrm{d}x + \int_{\pi}^{2\pi} (2\pi - x)e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} xe^{-ikx}\,\mathrm{d}x - \int_{\pi}^{2\pi} xe^{-ikx}\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\pi e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\left[x\frac{e^{-ikx}}{-ik}\right]_0^{\pi} - \int_0^{\pi}\frac{e^{-ikx}}{-ik}\,\mathrm{d}x - \left(\left[x\frac{e^{-ikx}}{-ik}\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\frac{e^{-ikx}}{-ik}\,\mathrm{d}x\right) + 2\pi\int_{\pi}^{2\pi} e^{-ikx}\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\frac{\pi}{-ik}e^{-ik\pi} - 0 - \left[\frac{e^{-ikx}}{(-ik)^2}\right]_0^{\pi} - \left(\frac{2\pi}{-ik}e^{-ik2\pi} - \frac{\pi}{-ik}e^{-ik\pi} - \left[\frac{e^{-ikx}}{(-ik)^2}\right]_{\pi}^{2\pi}\right) + 2\pi\left[\frac{e^{-ikx}}{-ik}\right]_{\pi}^{2\pi}\right) \\
&= \frac{1}{2\pi}\Bigg(\frac{\pi}{-ik}e^{-ik\pi} - \left(\frac{e^{-ik\pi}}{(-ik)^2} - \frac{1}{(-ik)^2}\right) \\
&\qquad - \left(\frac{2\pi}{-ik}e^{-ik2\pi} - \frac{\pi}{-ik}e^{-ik\pi} - \left(\frac{e^{-ik2\pi}}{(-ik)^2} - \frac{e^{-ik\pi}}{(-ik)^2}\right)\right) \\
&\qquad + 2\pi\left(\frac{e^{-ik2\pi}}{-ik} - \frac{e^{-ik\pi}}{-ik}\right)\Bigg) \\
&= \frac{1}{2\pi}\left(\frac{(-1)^k\pi}{-ik} - \frac{(-1)^k}{-k^2} + \frac{1}{-k^2} - \frac{2\pi}{-ik} + \frac{(-1)^k\pi}{-ik} + \frac{1}{-k^2} - \frac{(-1)^k}{-k^2} + \frac{2\pi}{-ik} - \frac{(-1)^k 2\pi}{-ik}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{-ik}\left((-1)^k\pi - 2\pi + (-1)^k\pi + 2\pi - (-1)^k 2\pi\right) + \frac{1}{-k^2}\left(-(-1)^k + 1 + 1 - (-1)^k\right)\right) \\
&= \frac{1}{2\pi}\left(\frac{1\cdot 0}{-ik} + \frac{2(1-(-1)^k)}{-k^2}\right) \\
&= \frac{1-(-1)^k}{-\pi k^2} \\
&= \frac{(-1)^k - 1}{\pi k^2}
\end{aligned}
$$

Fourier series:

$$F(x) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{k^2}\left((-1)^k - 1\right)e^{ikx}$$

We could derive the real-valued Fourier series analogously to the square wave, since we have a factor $\frac{1}{k^2}$ no sign should appear and like with the dirac we would end up with a sum of cosines which is obvious as the function is even.

There is, however, a much simpler way (that we also could have used to get the complex coefficients c_k). The Dirac delta, the square wave and the repeating ramp can be obtained by integrating and differentiating, respectively. Integrating the Dirac delta gives the square wave divided by 2[1]; integrating the square wave gives the ramp. We can use this insight to easily derive the real-valued Fourier series of the Dirac delta by differentiating the square pulse (and adding the factor 2). We can further derive the real-valued series for the ramp function from the square pulse by integrating. The only piece missing is the integration constant.

$$
\begin{aligned}
DD: &\quad \frac{2}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \cos(kx) \\
SW: &\quad \frac{4}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{\sin(kx)}{k} \\
RR: &\quad c_0 - \frac{4}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{\cos(kx)}{k^2}
\end{aligned}
$$

The meaning of the constant c_0 gives us its value: It is the average value of the function over one period, so $c_0 = \pi/2$ for the ramp.

If we had to truncate a series, then an interesting question is how many terms we need to achieve a certain accuracy. We can answer this by looking at the decay rate of the coefficients. Integration makes functions smoother; higher frequencies become less important. This result translates to the discrete Fourier transform in a very similar manner.

function	**decay rate**
delta	1
square wave	$1/k$
ramp	$1/k^2$

Exercise 2: Simple JPEG Encoder

See the attached Python code. We created two lookup tables, one for the cosine values and one for the combination of coefficients. The DCT routine of the sample solution is very basic – the four nested loops take $\mathcal{O}(n^4)$.

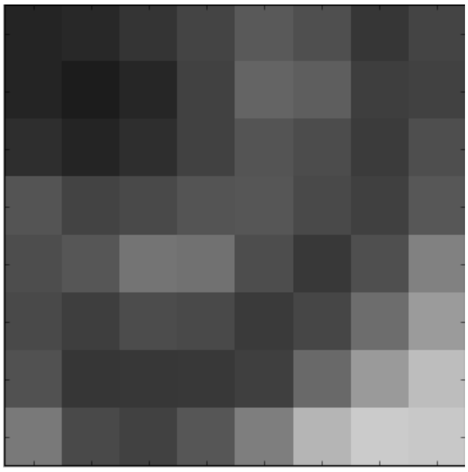
F_{00} is the average value. Speaking of images, this gives the background colour of an entire 8-by-8 block. Image processing refers to F_{00} as the DC coefficient. Coefficient F_{uv} far away from the DC coefficient contribute only minor information to the image. Quantisation reduces these coefficients.

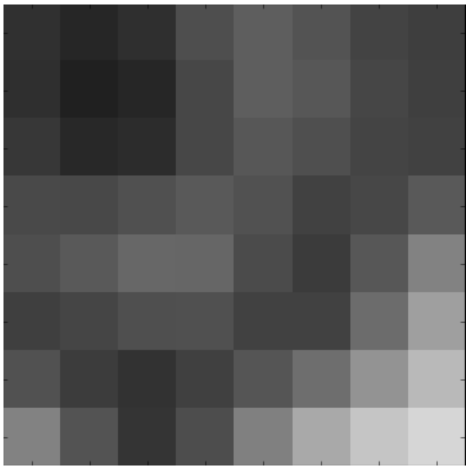
The IDCT is straight forward. In contrast to the lecture, this IDCT utilises equally distributed normalisation factors. The IDCT therefore only switches the roles of pixels and frequencies.

$$
f_{xy} = \frac{1}{\sqrt{2N}} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} c_u c_v F_{uv} \cos\left(\frac{(2x+1)u\pi}{2N}\right) \cos\left(\frac{(2y+1)v\pi}{2N}\right)
$$

[1]The dirac spike is in some sense a jump of size one at a discontinuities, in our case we want to do a jump of size 2 at the discontinuities of the square wave so we need to add a factor 2

The original image is shown on the left; the one after encoding and decoding with JPEG is shown on the right.





The reconstructed image matches the original one pretty well, but appears smoother. Higher frequencies are quantised by the high values of the quantisation matrix, which results in a coarser approximation. The quantisation step of JPEG is nevertheless reasonable because the human visual system is less sensitive for high frequencies.

Exercise 3: Discrete Cosine Transform

a) Show that the corresponding Fourier coefficients are real

$$F_k = \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \tag{1}$$

The proof is done in the following steps:

① Isolate the symmetry condition

② Insert the symmetry condition

③ Assemble terms to a sum over f_n

④ Make terms ”real”

$$
\begin{aligned}
F_k &= \frac{1}{2N} \sum_{n=-N+1}^{N} f_n \omega_{2N}^{-kn} \\
&\overset{①}{=} \frac{1}{2N} \left(\sum_{n=-N+1}^{-1} f_n \omega_{2N}^{-kn} + f_0 \omega_{2N}^{0} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N \omega_{2N}^{-kN} \right) \\
&= \frac{1}{2N} \left(\sum_{n=1}^{N-1} f_{-n} \omega_{2N}^{kn} + f_0 e^{0} + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N e^{-i2\pi kN/2N} \right) \\
&\overset{②}{=} \frac{1}{2N} \left(\sum_{n=1}^{N-1} f_n \omega_{2N}^{kn} + f_0 + \sum_{n=1}^{N-1} f_n \omega_{2N}^{-kn} + f_N e^{-i\pi k} \right) \\
&\overset{③}{=} \frac{1}{2N} \left(f_0 + \sum_{n=1}^{N-1} f_n \underbrace{\left(\omega_{2N}^{kn} + \omega_{2N}^{-kn} \right)}_{=\omega_{2N}^{kn} + \left(\omega_{2N}^{kn}\right)^* = 2\mathrm{Re}\{\omega_{2N}^{kn}\}} + f_N e^{-i\pi k} \right) \\
&\overset{④}{=} \frac{1}{2N} \left(f_0 + 2 \sum_{n=1}^{N-1} f_n \underbrace{\mathrm{Re}\{e^{i2\pi kn/2N}\}}_{=\mathrm{Re}\{\cos\left(\frac{\pi kn}{N}\right) + i \sin\left(\frac{\pi kn}{N}\right)\}} + f_N \left(\cos(-\pi k) + i \sin(-\pi k) \right) \right) \\
&= \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi kn}{N} \right) + \frac{1}{2} f_N \cos(\pi k) \right) \in \mathbb{R} \qquad \text{q.e.d.}
\end{aligned}
$$

b) Show that the F_k also have a symmetry:

Due to $\cos(x) = \cos(-x)$ we obtain easily:

$$
\begin{aligned}
F_{-k} &= \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{-\pi kn}{N} \right) + \frac{1}{2} f_N \cos(-\pi k) \right) \\
&= \frac{1}{N} \left(\frac{1}{2} f_0 + \sum_{n=1}^{N-1} f_n \cos\left(\frac{\pi kn}{N} \right) + \frac{1}{2} f_N \cos(\pi k) \right) \\
&= F_k
\end{aligned}
$$

Since all $F_{-k} = F_k$, we need the F_k only for $k = 0, \ldots, N$ for a Cosine Transform.

c) Algorithm for the Cosine Transform

The procedure `FFT(f,N)` computes the correct coefficients, if we pass the $N + 1$ data from field g as a dataset of length $2N$ with symmetry $f_{-n} = f_n$.

From equation (1) of the worksheet we know that `FFT(f,N)` gets a dataset `f` with indices $n = -N + 1, \ldots, N$. We only have to compute the F_k for $k = 0, \ldots, N$.

So, the algorithm looks like this:

1. Set `f[0] := g[0]` $= f_0$
   For all $n = 1, \ldots, N - 1$:

Set `f[n] := g[n]` $= f_n$
Set `f[-n] := g[n]` $= f_n$
Set `f[N] := g[N]` $= f_N$

2. Call `FFT(f,`N`)`

3. *(Now the Fourier coefficients F_k are stored in the field f)*
For all $k = 0, \ldots, N$:
Set `g[k] := f[k]` $= F_k$