

# Algorithms of Scientific Computing (Algorithmen des Wissenschaftlichen Rechnens)

## Hierarchical Basis and Finite Element Method

### Exercise 1: Interpolation with Hierarchical Basis

Given a set of  $m$  data points

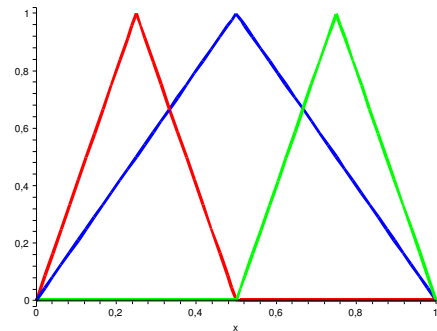
$$\{(x_1, u_1), \dots, (x_m, u_m)\},$$

where  $u_i = f(x_i), \forall i = 1, \dots, m$ .

Interpolate/approximate the underlying function  $f \in \mathcal{V}$  with  $f_n \in \mathcal{V}_n$  of the form

$$f(x) \approx f_n(x) = \sum_{k=1}^n c_k \psi_k(x)$$

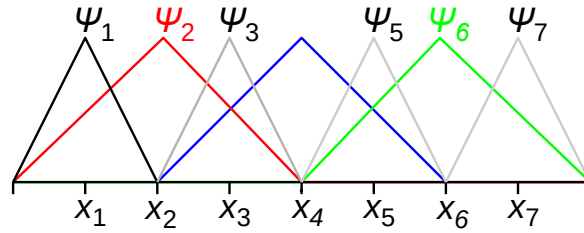
where  $\{\psi_k(x)\}$  are hierarchical basis as shown in figure.



- (a) Construct a linear system of equations to obtain  $c_k$  for  $n = 3$ .
- (b) Derive formulas for  $c_k$  (dependent on  $f_n$ ).
- (c) Repeat tasks (a) and (b) for  $n = 7$ .
- (d) Extend the interpolation to arbitrary data points  $n = 2^L - 1$ .

## Exercise 2: Interpolation with Partial-hierarchical Basis

Consider the following partial hierarchical basis:



Again, solve the interpolation problem:

- Construct a linear system of equations to obtain  $c_k$  for  $n = 7$ .
- Extend the system to  $n = 2^L - 1$ .
- How does this extend to solving the interpolation problem for a hierarchical basis?

## Exercise 3: Function approximation

In problems of function interpolation or regression, we are given a set of data points  $\{(x_1, y_1), \dots, (x_m, y_m)\}$ , we seek for a function  $f \in \mathbb{V}$ , such that the distance between this function and all data points are minimized, e.g.,

$$f(x) = \arg \min_f \left\{ \sum_{j=1}^m \|y_j - f(x_j)\| \right\}. \quad (1)$$

We can express  $f$  in the form of

$$f(x) = \sum_i \alpha_i \cdot \phi_i(x), \quad (2)$$

where  $\{\phi_i\}$  is a set of known basis functions that span the function space, i.e.,  $\mathbb{V} = \text{span}\{\phi_i\}$ . Now the problem of searching for  $f$  has become searching for  $\{\alpha_i\}$ , the coefficients of basis functions.

In this exercise, we would like to employ the same strategy to approximate a function  $g(x)$ . The goal is to look for an  $f$  of the form (2) that can best approximate  $g(x)$ , e.g.,

$$f(x) = \arg \min_f \|g(x) - f(x)\|_2. \quad (3)$$

- Show that solving (3) is equivalent to finding a set of coefficients  $\{\alpha_k\}$ , where

$$\int_{-\infty}^{+\infty} \phi_k(x) \left( g(x) - \sum_i \alpha_i \cdot \phi_i(x) \right) dx \stackrel{!}{=} 0, \quad \forall k \quad (4)$$

- Transform (4) into a linear system of the form  $A\alpha = b$ . What does  $A$  look like?
- Let  $g(x) = -4x^2 + 4x$ , where  $x \in [0, 1]$ . Solve (4) with nodal basis hat functions and piecewise constant functions.