Algorithms of Scientific Computing
(Algorithmen des Wissenschaftlichen Rechnens)
Multi-dimensional Quadrature

Exercise 1: 2D Basis function

Try to understand the ipython notebook implementation in visualizePagoda.ipynb that plots the 2D hat functions – the pagoda base – for arbitrary subspaces. Try to vary the xlevel and ylevel to see the different hierarchical subspaces for different levelvectors.

Optional: Define surplus values and plot the hyperplane that is generated by adding up all subspaces of a regular grid. Do not forget to scale the hats by their corresponding surplus values.

Exercise 2: Multidimensional Quadrature

In order to find an alternative way to approximate the d-dimensional integral

$$F_d := \int_{[0,1]^d} f(x_1, \ldots, x_d) \, d(x_1, \ldots, x_d)$$

we use a regular d-dimensional grid of step size $h = 2^{-l}$ stored in an array with indices in $[0, 2^l]^d$.

We don't want to be bothered with special boundary treatment so we simplify this task assuming our function is 0 on the boundary, i.e.

$$\forall \mathbf{x} = [x_1, ..., x_d]^T : \exists j \text{ such that } (x_j = 0 \vee x_j = 1) \Rightarrow f(\mathbf{x}) = 0$$

In the next step we compute the d-dimensional hierarchical surpluses:

---

loop $j = 1, \ldots, d$ **over the dimensions**
    loop over all 1d subgrids discretizing spatial direction j
    (fix all coordinates except x_j, what you'll get is a 1d array
    of grid points like on worksheet 4 — for $d = 2$ this comes down
    to processing all rows and all columns once)
        Compute 1d surpluses in dimension j **(in place) for each subarray**

Once we have the surpluses $v_{\underline{l},\underline{i}}$ (indexing with level $\underline{l}$ and point index $\underline{i}$ as introduced in the lecture) we only need to multiply them with the base area of the associated pagoda and sum everything up. The base area of the pagoda associated with grid point $x_{\underline{l},\underline{i}}$ is $2^{-|\underline{l}|_1}$.

Let

$$f(x_1, x_2) = 16x_1(1 - x_1)x_2(1 - x_2) \; = \; \left(1 - 4(x_1 - \frac{1}{2})^2\right) \cdot \left(1 - 4(x_2 - \frac{1}{2})^2\right).$$

1. Implement this method!
   For $d = 2$ and $l = 3$ this can be done easily in a spreadsheet (using copy and paste is not only ok but also encouraged for thorough understanding).

2. Calculate all pagoda volumes.

3. Draw all 81 grid points in a 2D grid and mark grid points with the same values for their associated pagoda volume.

4. Draw a new grid where you exclude points for which the associated pagoda volume is below a certain threshold (try e.g. 0.1, 0.01, 0.001 and 0.0001). Do you observe a pattern?