Technische Universität München
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Scientific Computing in Computer Science

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Worksheet
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Fundamental Algorithms 1

– Solution Examples –

Exercise 1

Prove (by induction over n) that $\frac{1}{3}n^2 + 5n + 30 \in O(n^2)$ for all $n \in \mathbb{N}^+$.

Solution:

(Note: we wouldn't have to prove by induction, but it's a simple case to practice it.)

$$f := \frac{1}{3}n^2 + 5n + 30 \in O(n^2) \qquad \Leftrightarrow \qquad \exists c > 0 \exists n_0 \forall n \geq n_0 : f(n) \leq cn^2$$

Let $c := 100$, $n_0 := 1$.

Base case: $n = n_0 = 1$

$$\frac{1}{3} + 5 + 30 = 35\frac{1}{3} \leq 100$$

Induction hypothesis: For some $n \in \mathbb{N}$: $f(n) \leq 100n^2$

Inductive step:

$$\begin{aligned}
f(n+1) &= \frac{1}{3}(n+1)^2 + 5(n+1) + 30 \\
&= \frac{1}{3}(n^2 + 2n + 1) + 5(n+1) + 30 \\
&= f(n) + \frac{2}{3}n + \frac{16}{3} \\
&\overset{ih}{\leq} 100n^2 + \frac{2}{3}n + \frac{16}{3} \\
&\leq 100n^2 + 200n + 100 \\
&= 100(n+1)^2
\end{aligned}$$

q.e.d.

Note: we chose a pretty large c for this prove – you should re-do this proof with smaller values for c (such as $c = 1$) and see what happens.

Exercise 2

(a) Compare the growth of the following functions using the o-, O-, and Θ-notation:

1. $n \log n$
2. n^l for all $l \in \mathbb{N}$
3. 2^n

(b) Try to give a simple characterization of the growth of the following expressions using the Θ-notation:

$$1)\ \sum_{i=1}^{n} \frac{1}{i} \qquad\qquad 2)\ \log(n!)$$

Hint for $\log(n!)$: try to prove $n^{\frac{n}{2}} \leq n! \leq n^n$ first!

Solution:

(a) $n^l \in o(2^n)$ for all $l \in \mathbb{N}$, because by de l'Hôpital's rule:

$$\lim_{n\to\infty} \frac{n^l}{2^n} = \lim_{n\to\infty} \frac{l \cdot n^{l-1}}{2^n \cdot \ln 2} = \lim_{n\to\infty} \frac{l \cdot (l-1) \cdot n^{l-2}}{2^n \cdot (\ln 2)^2} = \ldots = \lim_{n\to\infty} \frac{l!}{2^n \cdot (\ln 2)^l} = 0$$

Therefore, $n^l \in O(2^n)$ for all $l \in \mathbb{N}$.

Obviously, $n^1 \in o(n \log n)$ and $n^1 \in O(n \log n)$, but for $l \geq 2$:

$$\lim_{n\to\infty} \frac{n \ln n}{n^l} = \lim_{n\to\infty} \frac{\ln n}{n^{l-1}} = \lim_{n\to\infty} \frac{1}{n \cdot (l-1) \cdot n^{l-2}} = 0$$

Therefore $n^l \in \omega(n \log n)$ for all $l \geq 2$. This also holds for any real $l > 1$.

As a consequence, $n \log n \in o(2^n)$.

(b) 1) $\displaystyle\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\ln n)$:

Consider the functions $u(x) := \frac{1}{\lfloor x \rfloor}$ and $l(x) := \frac{1}{\lceil x \rceil}$, then:

$$l(x) \leq \frac{1}{x} \leq u(x) \quad \Rightarrow \quad \int_1^n l(x)\,dx \leq \int_1^n \frac{1}{x}\,dx \leq \int_1^n u(x)\,dx$$

$$\Rightarrow \quad \sum_{i=2}^{n} \frac{1}{i} \leq \ln n - \ln 1 \leq \sum_{i=1}^{n-1} \frac{1}{i}$$

(draw a graph of $u(x)$ and $l(x)$ to see why the integrals are given by these sums).

Thus, $\ln n \leq \sum_{i=1}^{n-1} \frac{1}{i} \leq \sum_{i=1}^{n} \frac{1}{i}$, and therefore $\ln n \in O\left(\sum_{i=1}^{n} \frac{1}{i}\right)$.

As $2 \cdot \sum_{i=2}^{n} \frac{1}{i} = 2 \cdot \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) > 1$, we know that

$$3\sum_{i=2}^{n} \frac{1}{i} = 2\sum_{i=2}^{n} \frac{1}{i} + \sum_{i=2}^{n} \frac{1}{i} > 1 + \sum_{i=2}^{n} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i},$$

and, therefore,

$$\sum_{i=1}^{n} \frac{1}{i} < 3 \sum_{i=2}^{n} \frac{1}{i} \leq 3 \ln n \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{1}{i} \in O(\ln n), \quad \text{q.e.d.}$$

2) Using $n^{\frac{n}{2}} \leq n! \leq n^n$, we get:

$$\ln n^{\frac{n}{2}} \leq \ln(n!) \leq \ln n^n \quad \Rightarrow \quad \frac{n}{2} \ln n \leq \ln(n!) \leq n \ln n,$$

which leads directly to the result $\ln(n!) \in \Theta(n \ln n)$.

Proof for $n^{\frac{n}{2}} \leq n! \leq n^n$:

It is obvious that $n! = 1 \cdot 2 \cdot \ldots \cdot n \leq n \cdot n \cdot \ldots \cdot n = n^n$.
To prove $n^{\frac{n}{2}} \leq n!$, or $n^n \leq (n!)^2$, we show that $\frac{(n!)^2}{n^n} \geq 1$:

$$\frac{(n!)^2}{n^n} = \frac{n!}{n^n} \cdot n! = \prod_{i=0}^{n-1} \frac{n-i}{n} \cdot \prod_{i=0}^{n-1} (i+1) = \prod_{i=0}^{n-1} \frac{(n-i)(i+1)}{n}$$

and $(n-i)(i+1) = -i^2 + ni - i + n = n + i(n-1-i) \geq n$. Therefore, all factors of the product are ≥ 1. Consequently, the product itself is ≥ 1.

Exercise 3

Let $l(x)$ be the number of bits of the representation of x in the binary system. Prove:

$$\sum_{i=1}^{n} l(i) \in \Theta(n \log n)$$

Solution:

We need the following equalities:

- $\sum_{i=1}^{n} \log i = \log \left(\prod_{i=1}^{n} i \right) = \log(n!) \in \Theta(n \log n)$, (see exercise 1(b), part 2!), and
- $l(i) = \lfloor \log_2 i \rfloor + 1$ (if the binary representation of a number has l bits, the respective number i will be between 2^{l-1} and $2^l - 1$).

If we can show that

$$c_1 \log_2 i \leq \lfloor \log_2 i \rfloor \leq \log_2 i$$

for some constant $0 < c_1 < 1$ (the second inequality is a trivial result of the definition of $\lfloor \rfloor$), and use the transformation

$$\sum_{i=1}^{n} l(i) = \sum_{i=1}^{n} (\lfloor \log_2 i \rfloor + 1) = n + \sum_{i=1}^{n} \lfloor \log_2 i \rfloor,$$

we get

$$c_1 \left(n + \sum_{i=1}^{n} \log_2 i \right) \leq \sum_{i=1}^{n} l(i) \leq n + \sum_{i=1}^{n} \log_2 i \quad \Rightarrow \quad \sum_{i=1}^{n} l(i) \in \Theta(n \log n)$$

We still have to prove that $c_1 \log_2 i \leq \lfloor \log_2 i \rfloor$ for some c_1:
For $i \geq 3$, we can choose c_1, such that $i^{c_1} < \frac{i}{2}$ (choose $c_1 := \frac{1}{4}$, e.g.). Then

$$c_1 \log_2 i = \log_2 \left(i^{c_1}\right) < \log_2 \frac{i}{2} = \log_2 i - 1 < \lfloor \log_2 i \rfloor .$$

As the inequality is also correct for $i \in \{1, 2\}$, we are finished.

Exercise 4

Prove that Θ defines an equivalence relation on the set of functions $\{f \mid f : \mathbb{N} \to \mathbb{R}\}$. Use that $(f, g) \in \Theta \Leftrightarrow f \in \Theta(g)$

Solution:

We define the relation Θ by $(f, g) \in \Theta :\Leftrightarrow f \in \Theta(g)$.
To show that Θ is an equivalence relation, we have to prove that:

- Θ is **reflexive**:
  as $f \in \Theta(f)$ (e.g., choose constants $c_1 := \frac{1}{2}$, and $c_2 := \frac{3}{2}$), by definition $(f, f) \in \Theta$;
- Θ is **symmetric**:
  if $f \in \Theta(g)$, then
  - $f \in O(g) \Rightarrow g \in \Omega(f)$
  - $f \in \Omega(g) \Rightarrow g \in O(f)$

  Therefore, by definition $g \in \Theta(f)$;
- Θ is **transitive**:
  if $f \in \Theta(g)$, and $g \in \Theta(h)$, then, there are constants c_1, c_2, c_3, and c_4, such that for sufficiently large n
  - $c_1 f(n) \leq g(n) \leq c_2 f(n)$
  - $c_3 g(n) \leq h(n) \leq c_4 g(n)$

  Therefore, $c_1 c_3 f(n) \leq h(n) \leq c_2 c_4 h(n)$ which leads to $f \in \Theta(h)$.