Fundamental Algorithms 3

– Solution Examples –

Exercise 1

Consider a partitioning algorithm that, in the worst case, will partition an array of m elements into two partitions of size $\lfloor \epsilon m \rfloor$ and $\lceil (1-\epsilon)m \rceil$, where ϵ is fixed, and $0 < \epsilon < 1$. Show that a quicksort algorithm based on this partitioning has a worst-case complexity of $O(n \log n)$.

Solution:

Again, we will only count comparisons between array elements.

Using that the partitioning step will require at most n comparisons, we get the following recurrence for the necessary number $C(n)$ of comparisons:

$$\begin{aligned} C(1) &= 0 \\ C(n) &= C(\epsilon n) + C((1-\epsilon)n) + n \end{aligned}$$

We guess $C(n) := an \log_2 n + b$ as the solution, and try to find constants a and b such that the recurrence is satisfied:
case $n = 1$**:**

$$C(1) = a \cdot 1 \cdot \log_2 1 + b = 0 \quad \Leftrightarrow b = 0,$$

hence, $C(n) = an \log_2 n$.

case $n > 1$**:** We insert our guess into the recurrence:

$$\begin{aligned} an \log_2 n = C(n) &= C(\epsilon n) + C((1-\epsilon)n) + n \\ \Leftrightarrow \quad an \log_2 n &= a\epsilon n \log_2(\epsilon n) + a(1-\epsilon)n \log_2((1-\epsilon)n) + n \\ \Leftrightarrow \quad an \log_2 n &= a\epsilon n \left(\log_2 \epsilon + \log_2 n\right) + a(1-\epsilon)n \left(\log_2(1-\epsilon) + \log_2 n\right) + n \\ \Leftrightarrow \quad an \log_2 n &= a\epsilon n \log_2 \epsilon + a\epsilon n \log_2 n + \\ &\quad\; a(1-\epsilon)n \log_2(1-\epsilon) + a(1-\epsilon)n \log_2 n + n \\ \Leftrightarrow \quad an \log_2 n &= a\epsilon n \log_2 \epsilon + a\epsilon n \log_2 n + \end{aligned}$$

$$
\begin{array}{rcl}
& & an\log_2(1-\epsilon) - a\epsilon n\log_2(1-\epsilon) + an\log_2 n - a\epsilon n\log_2 n + n \\
\Leftrightarrow \quad 0 & = & a\epsilon n\log_2\epsilon + an\log_2(1-\epsilon) - a\epsilon n\log_2(1-\epsilon) + n \\
\Leftrightarrow \quad 0 & = & an\left(\epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon)\right) + n \\
\Leftrightarrow \quad a & = & \dfrac{-1}{\epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon)}
\end{array}
$$

Thus, the recurrence is satisfied if

$$C(n) = \frac{-n\log_2 n}{\epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon)}$$

Note that the constant a will be very large for values of ϵ that are close to either 0 or 1. Thus, even very bad partitions will not destroy the $O(n\log n)$ complexity, provided that the respective partition sizes are bounded by ϵn and $(1-\epsilon)n$. However, bad partitions will still lead to slow algorithms due to the large constant factor involved.

K-Exercise 2 (An Iterative MergeSort)

The following iterative implementation of the MergeSort algorithm is proposed:

```
ItMergeSort(A: Array[0..n-1]) {
   // n assumed to be a power of 2: n=2^k
   k := log2(n)
   //
   m := 2
   for L from 1 to k do {
      for i from 0 to (n/m)-1 do {
         MergeIP(A[i*m .. i*m+(m/2-1],
                 A[i*m+(m/2) .. i*m+(m-1],
                 A[i*m .. i*m+(m-1));
      };
      m := 2*m;
   };
}
```

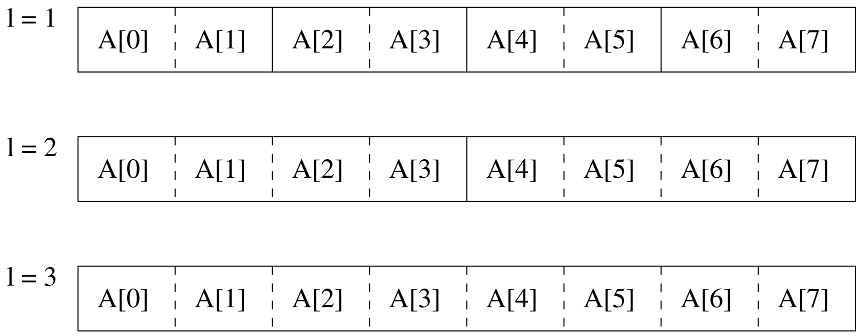
The procedure MergeIP is equivalent to the procedure Merge discussed in the lecture, but can work directly on the array A (i.e., merges two adjacent subarrays of A).

a) Describe shortly and in plain words, how ItMergeSort compares to the recursive MergeSort implementation discussed in the lecture. For that purpose, draw a diagram that illustrates the sorting of an array A[0..7] for ItMergeSort.

b) Formulate a loop invariant for the L-loop of the algorithm, and prove its correctness.

Solution:

a) In each iteration of the L-loop two adjacent subarrays are merged. The lengths of the merged subarrays ($m/2$) is doubled from each L-loop iteration to the next. In that way, the *same* merging steps as for the recursive implementation of MergeSort are executed. The divide steps are implicitly performed on the array.



b) We propose the following loop invariant:

> At entry of the L-loop, the array A consists of $\frac{2n}{m}$ subarrays of length $\frac{m}{2}$, where $m = 2^L$. Each of the subarrays is sorted.

Here's a sketch of the proof:

Initialisation: on the first entry, for $L = 1$ and $m = 2^1$, the length of the subarrays is claimed to be $\frac{m}{2} = 1$ with $\frac{2n}{2} = n$ subarrays – this is obviously satisfied, as subarrays of length 1 are always sorted.

Maintenance: The i-loop will take $\frac{n}{m}$ pairs of two adjacent subarrays and merge them using the procedure MergeIP. Provided the correctness of MergeIP, this will lead to $\frac{n}{m}$ subarrays of twice the length, which satisfies the loop invariant for the next iteration. Note that m is multiplied by 2, to retain $m = 2^L$.

Termination: At termination, $L = k + 1$ and thus $m = 2^{k+1} = 2n$. Hence, we have only $\frac{2n}{2n} = 1$ subarray of length $\frac{2n}{2} = n$, which is sorted. This implies the correctness of the sorting algorithm.