Fundamental Algorithms

Chapter 8: AVL Trees

Dirk Pflüger

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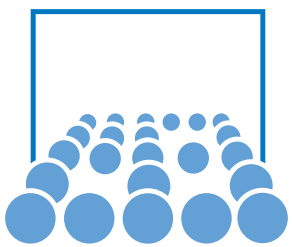


Part I

AVL Trees

Binary Search Trees – Summary

Complexity of Searching:

- worst-case complexity depends on height of the search trees
- $O(\log n)$ for balanced trees

Inserting and Deleting:

- insertion and deletion might change balance of trees
- question: how expensive is re-balancing?

Test: Inserting/Deleting into a (fully) balanced tree
$\Rightarrow$ strict balancing (uniform depth for all leaves) too strict

Height Balance

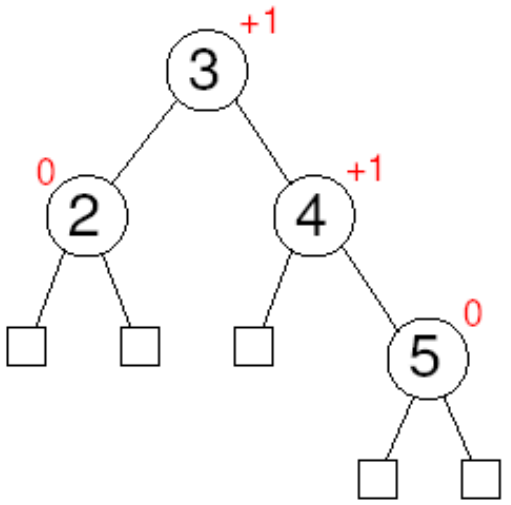
Definition (height balance)

Let $h(x)$ be the height of a binary tree x.
Then, the height balance b(x.key) of a node x.key is defined as

$$b(\text{x.key}) := h(\text{x.rightChild}) - h(\text{x.leftChild})$$

i.e. the difference of the heights of the two subtrees of x.key.

Example:



height balance

AVL Trees

Definition (AVL tree)

A binary search tree x is called an AVL tree, if:

1. $b(\text{x.key}) \in \{-1, 0, 1\}$, and
2. x.leftChild and x.rightChild are both AVL trees.

= the height balance of every node must be -1, 0, or 1

Number of Nodes in an AVL Tree

Crucial Question for Complexity:

What is the maximal and minimal number of nodes that can be stored in an AVL tree of given height h?

Maximal number:

- a full binary tree has a height balance of 0 for every node
- thus: full binary trees are AVL trees
- number of nodes: $2^h - 1$ nodes.

Minimal Number of Nodes in an AVL Tree

Minimal number:

A“minimal” AVL tree of height h consists of

- a root node
- one subtree that is a minimal AVL tree of height $h-1$
- one subtree that is a minimal AVL tree of height $h-2$

$\Rightarrow$ leads to recurrence:

$$N_{\text{minAVL}}(h) = 1 + N_{\text{minAVL}}(h-1) + N_{\text{minAVL}}(h-2)$$

In addition, we know that

- a minimal AVL tree of height 1 has 1 node: $N_{\text{minAVL}}(1) = 1$
- a minimal AVL tree of height 2 has 2 nodes: $N_{\text{minAVL}}(2) = 2$

Minimal Number of Nodes in an AVL Tree (2)

Solve recurrence:

$$
\begin{aligned}
N_{\text{minAVL}}(h) &= 1 + N_{\text{minAVL}}(h-1) + N_{\text{minAVL}}(h-2) \\
N_{\text{minAVL}}(1) &= 1 \\
N_{\text{minAVL}}(2) &= 2
\end{aligned}
$$

Compare with Fibonacci numbers: $f_n = f_{n-1} + f_{n-2}$, $f_0 = f_1 = 1$

h	1	2	3	4	5	6	7	8
f_h	1	2	3	5	8	13	21	34
$N_{\text{minAVL}}(h)$	1	2	4	7	12	20	33	54

Claim: $N_{\text{minAVL}}(h) = f_{h+1} - 1$ (proof by induction)

Minimal Number of Nodes in an AVL Tree (2)

Minimum number of nodes: $N_{\text{minAVL}}(h) = f_{h+1} - 1$

- inequality for Fibonacci numbers: $2^{\lfloor n/2 \rfloor} \leq f_n \leq 2^n$

$$\Rightarrow \quad 2^{\lfloor (h+1)/2 \rfloor} - 1 \leq N_{\text{minAVL}}(h) \leq 2^{h+1} - 1$$

- thus: an AVL tree that contains n nodes will be of height $\Theta(\log n)$

Corollaries:

- Searching in an AVL tree has a time complexity of $\Theta(\log n)$
- Inserting, or deleting a single element in an AVL tree has a time complexity of $\Theta(\log n)$
- BUT: standard inserting/deleting will probably destroy the AVL property.

Part II

Algorithms on AVL Trees

Inserting and Deleting on AVL Trees

General Concept:

- insert/delete via standard algorithms
- after insert/delete: load balance b(node) might be changed to $+2$ or -2 for certain nodes
- re-balance load after each step

Requirements:

- re-balancing must have $O(\log n)$ worst-case complexity

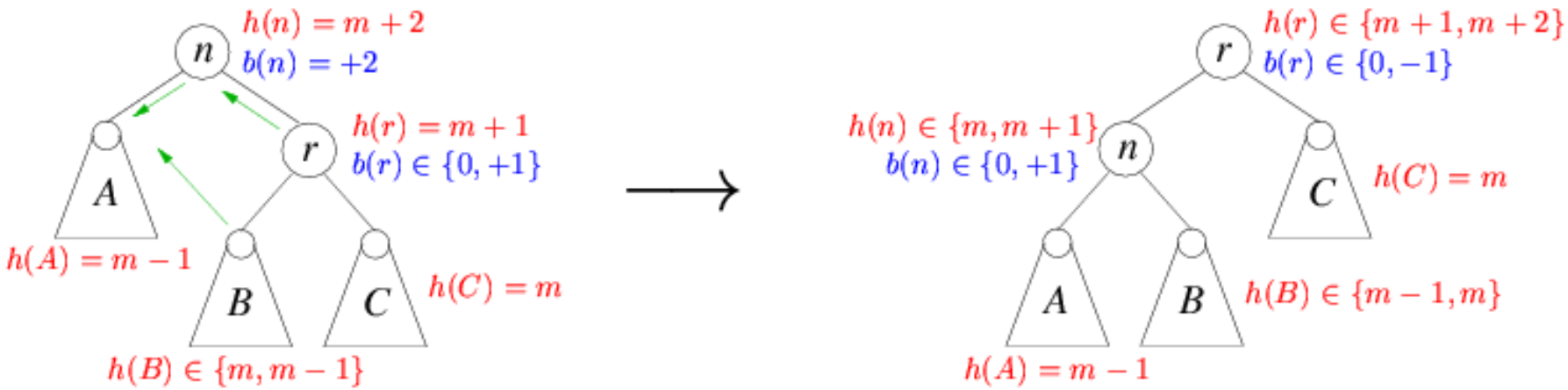
Solution:

- apply certain “rotation” operations

Left Rotation

Situation:

- height balance of the node is +2 (or larger), and
- height balance of the right subtree is 0, or +1



- can reduce height of the subtree and require further rotations

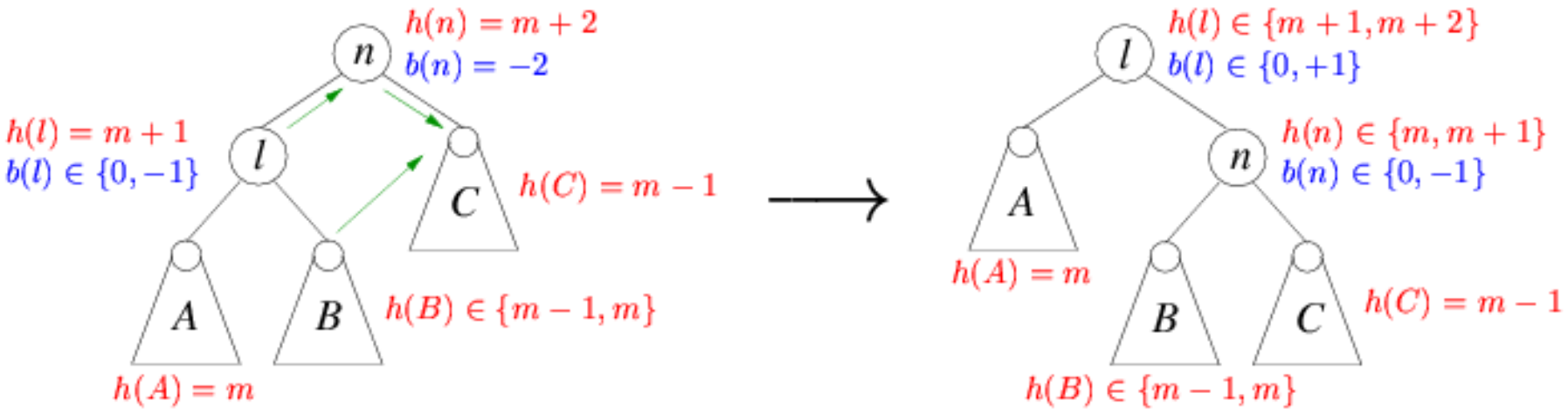
Left Rotation – Implementation

```
LeftRotAVL (x:BinTree) {

    x := ( x.rightChild.key,
           (x.key, x.leftChild , x.rightChild.leftChild),
           x.rightChild.rightChild
         );

}
```

Right Rotation

Situation:

- height balance of the node is -2 (or smaller), and
- height balance of the left subtree is 0, or -1



- can reduce height of the subtree and require further rotations

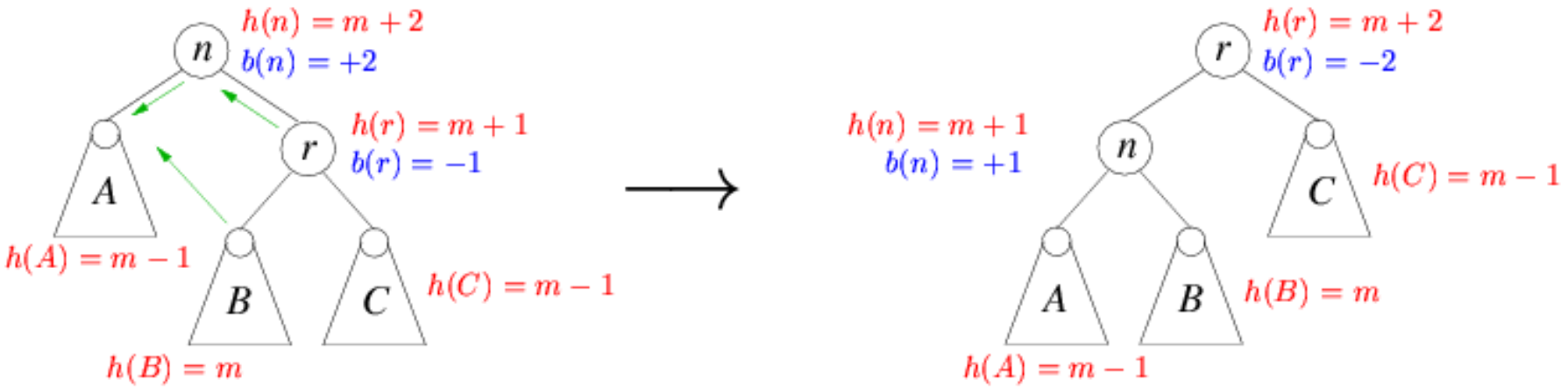
Right Rotation – Implementation

```
RightRotAVL (x:BinTree) {

        x := ( x.leftChild.key,
               x.leftChild.leftChild,
               (x.key, x.leftChild.rightChild, x.rightChild)
             );

}
```

Right-Left Rotation

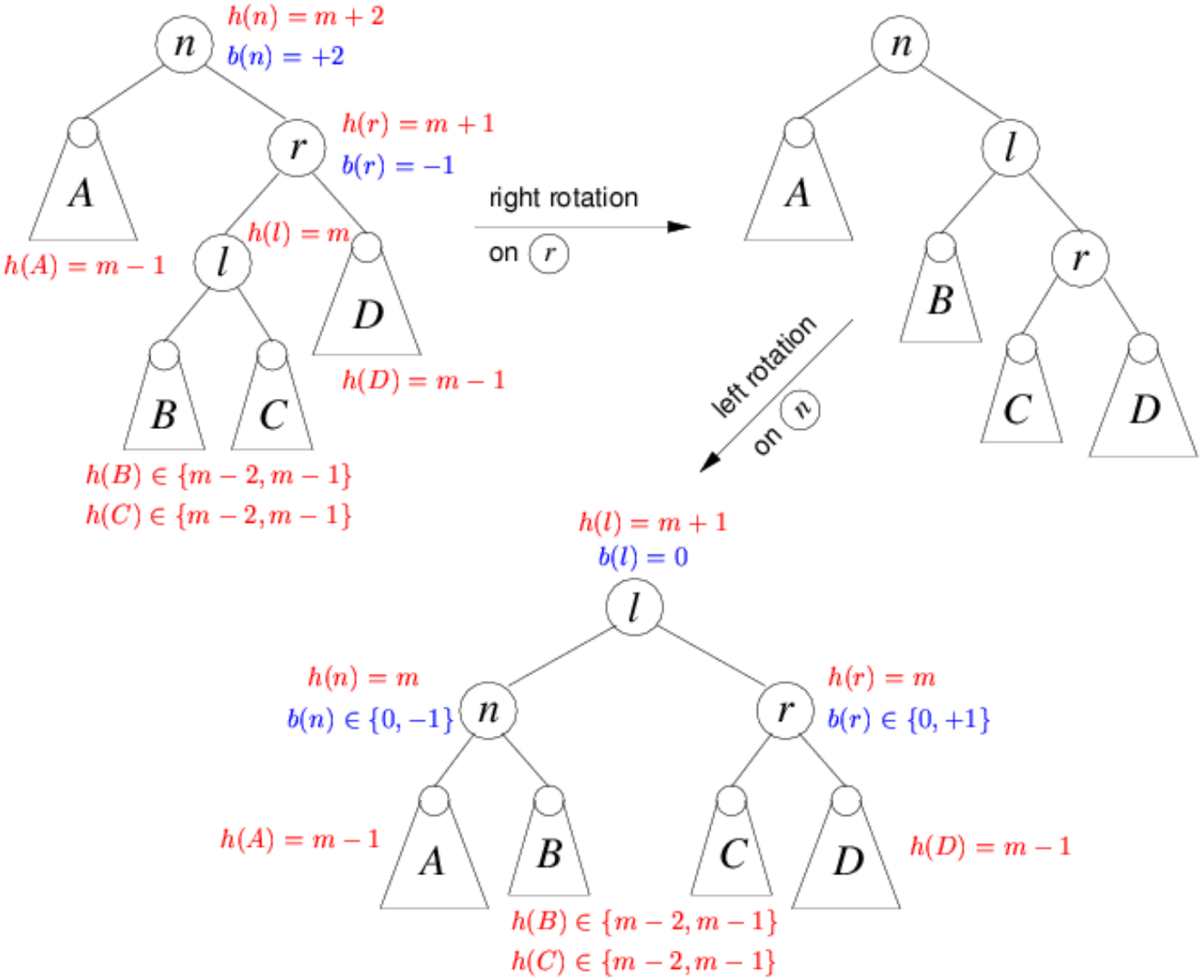
Situation:

- height balance of the node is +2 , and
- height balance of the right subtree is -1
  $\Rightarrow$ left rotation successful?



- left rotation is not sufficient to restore the AVL property

Solution: Two Successive Rotations



Right-Left Rotation – Implementation

```
RightRotLeftRot (x:BinTree) {
    RightRotAVL(x.rightChild);
    LeftRotAVL(x);
}
```

In a single procedure:

```
RightLeftRotAVL (x:BinTree) {
    x := ( x.rightChild.leftChild.key,
           ( x.key,
                x.leftChild,
                x.rightChild.leftChild.leftChild ),
           ( x.rightChild.key,
                x.rightChild.leftChild.rightChild,
                x.rightChild.rightChild)
         );
}
```

Left-Right Rotation

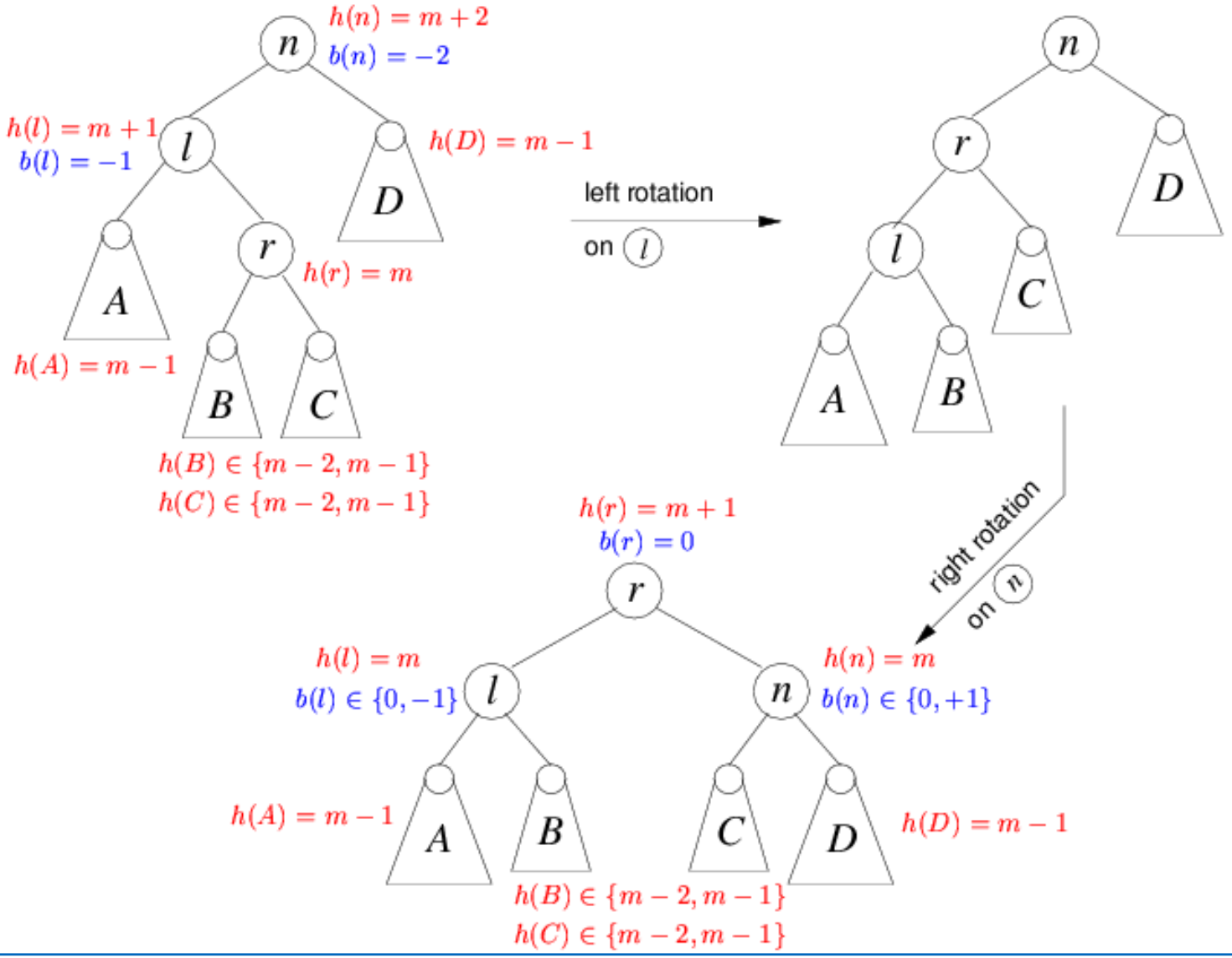
Situation:

- height balance of the node is -2 , and
- height balance of the right subtree is +1

Analogous to Right-Left Rotation:

- again, right rotation is not sufficient to restore the AVL property
- right rotation followed by a left rotation

Left-Right Rotation – Scheme



Left-Right Rotation – Implementation

```
LeftRotRightRot(x:BinTree) {
    LeftRotAVL(x.leftChild);
    RightRotAVL(x);
}
```

In a single procedure:

```
LeftRightRotAVL(x:BinTree) {
    x := ( x.leftChild.rightChild.key,
           ( x.leftChild.key,
                x.leftChild.leftChild,
                x.leftChild.rightChild.leftChild),
           ( x.key,
                x.leftChild.rightChild.rightChild,
                x.rightChild )
         );
}
```

Effect of Rotation Operations

Observations:

- LeftRightRot and RightLeftRot always reduce the height of the (sub-)tree by 1.
- LeftRotAVL and RightRotAVL reduce the height by at most 1 (might keep it unchanged)
- thus: AVL property can be violated in a parent node!

After inserting one node into an AVL tree:

- at most one rotation required to restore AVL property

Effect of Rotation Operations (2)

After inserting one node into an AVL tree:

- at most one rotation required to restore AVL property

After deleting one node of an AVL tree:

- up to log h rotations can be required to restore AVL property

For insertion and deletion:

- imbalance might occur on a much higher level
- thus: AVL property has to checked in the entire branch of the tree (up to the root)

Corollary:

Time complexity for **deleting**, **inserting**, and **searching** in an AVL tree is $O(\log n)$