Solving the Shallow Water Equations

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References

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  Finite Volume Methods for Hyperbolic Problems,
  Cambridge University Press, 6th edition, 2007

Motivation: Conservation laws

General form of conservation laws in 1D:

$$q_t(x,t) + f(q(x,t))_x = 0 \tag{1}$$

where:

- $x, t \in \mathbb{R}$: space and time variables
- $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$: conserved quantity vector (function over space and time)
- $f : \mathbb{R}^d \to \mathbb{R}^d$: flux function, simplest case: $f(q) = u \cdot q$ for a constant velocity $u \in \mathbb{R}$.

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Quasilinear form:

$$q_t(x,t) + f'(q(x,t))q_x = 0 \tag{2}$$

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Finite Volumes: subdivision of domain into *grid cells*, each denoted by a range $C_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$.

Integration over x for all cells C_i gives the integral form:

$$\frac{d}{dt}\int\limits_{C_i} q(x,t)dx = f(q(x_{i-\frac{1}{2}},t)) - f(q(x_{i+\frac{1}{2}},t)) \tag{3}$$

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Integration over t for all time steps t_n:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x,t_{n+1}) - q(x,t_n)dx = \int_{t_n}^{t_{n+1}} f(q(x_{i-\frac{1}{2}},t)) - f(q(x_{i+\frac{1}{2}},t))dt \tag{4}$$

Motivation: Conservation laws

Discretization:
Average q over space by $Q_i^n := \frac{1}{\triangle x} \int\limits_{C_i} q(x, t_n) dx$.

Average $f(q)$ over time by $F_{i+\frac{1}{2}}^n := \frac{1}{\triangle t} \int\limits_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}}, t)) dt$.

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$$\Rightarrow Q_i^{n+1} = Q_i^n - \frac{\triangle t}{\triangle x} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{5}$$

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Problem: Assume Q_i^n is known, how do we get $F_{i-\frac{1}{2}}^n$, $F_{i+\frac{1}{2}}^n$ in order to compute Q_i^{n+1}?

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$\Rightarrow$ Central difference term, unstable.

Upwind Method

Modification:

$$Q_i^{n+1} = \begin{cases} Q_i^n - \frac{\triangle t}{\triangle x}\left(f(Q_i^n) - f(Q_{i-1}^n)\right) & f'(Q_i^n) \geq 0 \\ Q_i^n - \frac{\triangle t}{\triangle x}\left(f(Q_{i+1}^n) - f(Q_i^n)\right) & f'(Q_i^n) < 0 \end{cases}$$

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Upwind method, works if $f' \geq 0$ or $f' < 0$. But what about systems of equations or if the sign of f' changes?

Lax-Friedrichs Method

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Another attempt:

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$\Rightarrow$ Lax-Friedrichs Method, stable but diffusive.

Godunov’s method for linear systems

Assumption: flux function f is linear.

1. Reconstruct piecewise constant function $\tilde{q}(x, t_n)$ from cell averages Q_i^n.
2. Evolve hyperbolic equation exactly to obtain $\tilde{q}(x, t_{n+1})$ a time $\triangle t$ later.
3. Average over $\tilde{q}(x, t_{n+1})$ to obtain Q_i^{n+1} again.

How does step 2 work?

Godunov's method for linear systems

How does step 2 work? Solve a *Riemann problem*:

Let $\tilde{q}(x,t) \in \mathbb{R}^d$ be a vector of d conservative variables for $x, t \in \mathbb{R}$. f linear $\Rightarrow f(\tilde{q}) = A \cdot \tilde{q}$ for a matrix $A \in \mathbb{R}^{d \times d}$.
Then $\tilde{q}_t + A \cdot \tilde{q}_x = 0$.
Assuming A is diagonalizable we can find a diagonal matrix of Eigenvalues Λ so that $A = R \Lambda R^{-1}$.
Set $w := R^{-1} q$ then

$$\begin{aligned} \tilde{q}_t + A \cdot \tilde{q}_x &= 0 \\ \Rightarrow w_t + \Lambda \cdot w_x &= 0 \end{aligned}$$

is a modified system with a diagonal matrix Λ.

Godunov's method for linear systems

Evolving the system from time step 0 to time step t using $w_t + \Lambda w_x = 0$:

1. $w(x, 0) = R^{-1} \cdot q(x, 0)$
2. $w_p(x, t) = w_p(x - \lambda_p t, 0)$ for each component p
3. $q(x, t) = R \cdot w(x, t)$

Godunov's method for linear systems

Note that since $\tilde{q}$ is piecewise constant, the solution of the Riemann problem on the interface is constant over time. We denote the value by $\mathring{Q}_{i-\frac{1}{2}} = \mathring{q}(Q_{i-1}, Q_i)$. This gives us a generalized flux formulation:

- Solve the Riemann problem on each interface $x_{i-\frac{1}{2}}$ to obtain $\mathring{q}(Q_{i-1}, Q_i)$.
- Define flux $F^n_{i-\frac{1}{2}} = f(\mathring{q}(Q_{i-1}, Q_i))$.
- Apply the flux differencing formula:

$$\Rightarrow Q_i^{n+1} = Q_i^n - \frac{\triangle t}{\triangle x}(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})$$

Godunov's method for nonlinear systems

Shallow Water Equations: nonlinear system of equations

Difference to linear systems: Riemann problem usually not solvable, but approximation is possible.

Goal: For each cell interface, find matrix $\hat{A} \in \mathbb{R}^{d \times d}$ such that the system

$$\hat{q}_t + \hat{A}_{i-\frac{1}{2}} \hat{q}_x = 0$$

approximates the original system

$$q_t + f'(q) q_x = 0$$

by $\hat{A}_{i-\frac{1}{2}} \to f'(\tilde{q})$ as $Q_{i-1}, Q_i \to \tilde{q}$.

Godunov’s method with Roe linearization

Useful assumption: A single wave with speed s connects Q_i and Q_{i+1}, so that $f(Q_i) - f(Q_{i-1}) = s \cdot (Q_i - Q_{i-1})$. Our approximation must meet
$\hat{A}_{i-\frac{1}{2}}(Q_i - Q_{i+1}) = s \cdot (Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1})$.

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Idea: $q(\xi) := Q_{i-1} + \xi \cdot (Q_i - Q_{i-1})$, then $f(Q_i) - f(Q_{i-1}) =$

$$\int_0^1 \frac{df(q(\xi))}{d\xi} d\xi = \int_0^1 f'(q(\xi))q'(\xi) d\xi = \left[\int_0^1 f'(q(\xi)) d\xi\right] (Q_i - Q_{i-1}).$$

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$$\Rightarrow f(Q_i) - f(Q_{i-1}) = \left[\int_0^1 f'(q(\xi))d\xi\right](Q_i - Q_{i-1})$$

So a suitable choice for $\hat{A}$ is $\int_0^1 f'(q(\xi))d\xi$.

Godunov's method with Roe linearization

Usually, instead of integrating over ξ a transformation $z(\xi)$ is used to integrate a path on, because the resulting matrix may not be diagonalizable.

$$f(Q_i) - f(Q_{i-1}) = \left[\int_0^1 f'(q(z(\xi)))d\xi \right] (Z_i - Z_{i-1})$$

where $Z_i = z(Q_i), Z_{i-1} = z(Q_{i-1})$ and

$$Q_i - Q_{i-1} = \left[\int_0^1 \frac{dq(z(\xi)))}{dz} d\xi \right] (Z_i - Z_{i-1})$$

Roe solver for the 1D Shallow Water Equations

Nonlinear system of equations (ignoring source terms):

$$\begin{bmatrix} h \\ hu \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = 0$$

With $q := \begin{bmatrix} h \\ hu \end{bmatrix}$ they can be described as follows:

$$q_t + \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + \frac{1}{2}gq_1^2 \end{bmatrix}_x = 0$$

Roe solver for the 1D Shallow Water Equations

Choose as parameter vector $z := \frac{1}{\sqrt{h}} q$. Leaving out intermediate steps we obtain:

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} \end{bmatrix}$$

as our Roe matrix with the arithmetic average $\bar{h}$ and the *Roe average* $\hat{u}$. Applying Godunov's method for the time step, we have a full numerical scheme now.