3.4.2 Householder Method

Define special orthogonal and simple matrices H called Householder matrices (compare Givens):

$$u \in IR^n, \|u\|_2 = 1: \qquad H = I - 2uu^T$$

H as rank-1 perturbation of the identity is symmetric, idempotent and orthogonal:

$$H^T = I - 2uu^T = H$$

$$H^T H = H^2 = (I - 2uu^T)(I - 2uu^T) =$$

$$= I - 2uu^T - 2uu^T + 4u\underbrace{u^T u}_{1}u^T = I$$

For complex problems:

orthogonal → unitary, symmetric → hermitian

Use H_1 with appropriate vector u_1 to eliminate first column of A

$$H_1 A = (I - 2u_1u_1^T)\begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix} = \begin{pmatrix} a_1 - 2(u_1^T a_1)u_1 & \cdots & * \end{pmatrix} = \begin{pmatrix} \alpha & * \\ 0 & * \\ \vdots & * \\ 0 & * \end{pmatrix}$$

αe_1

To satisfy this equation we have to find a vector u_1 of length 1 with $a_1 - 2(u_1^T a_1)u_1 = \alpha e_1$

Because H_1 is orthogonal it holds:

$$\|H_1 a_1\|_2 = \|a_1\|_2 = \|\alpha e_1\|_2 = |\alpha| \Rightarrow \alpha = \pm\|a_1\|_2$$, e.g. $\alpha = \|a_1\|_2$

$$u_1 = \frac{a_1 - \|a_1\|_2 e_1}{2(u_1^T a_1)} = \frac{a_1 - \|a_1\|_2 e_1}{\left\|a_1 - \|a_1\|_2 e_1\right\|_2}$$

Repeat for all columns of A:

$$H_1 A = H_1 A_1 = (I - 2u_1 u_1^T) A = \left(\begin{array}{c|ccc} \|a_1\|_2 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array}\right)$$

Apply the same procedure on A_2, n-1 x m-1 matrix.

$$\tilde{H}_2 A_2 = (I - 2\tilde{u}_2 \tilde{u}_2^T) A_2 = \left(\begin{array}{c|ccc} \alpha_2 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_3 & \\ 0 & & & \end{array}\right)$$

Extend $u_2 := \begin{pmatrix} 0 \\ \tilde{u}_2 \end{pmatrix}$, $H_2 := I - 2u_2 u_2^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{H}_2 & \\ 0 & & & \end{pmatrix}$

For column 1,2,…,m this gives Householder matrices $H_1,\dots,H_m$ with

$$\underbrace{H_m \cdots H_2 H_1}_{Q^T} \cdot A = H_m \cdots H_3 \cdot \begin{pmatrix} \alpha_1 & * & * & & * \\ 0 & \alpha_2 & * & & * \\ 0 & 0 & & & \\ \vdots & \vdots & & \boxed{A_3} & \\ 0 & 0 & & & \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = R$$

Hence: $A = QR$

$$Q := \left(H_m \cdots H_2 H_1\right)^T = H_1 H_2 \cdots H_m$$

Remark: For m=n:

$H_1,\dots,H_{m-1}$ is enough, because last column is scalar