

3.4.2 Householder Method

Define special orthogonal and simple matrices H called Householder matrices (compare Givens):

$$u \in \mathbb{R}^n, \|u\|_2 = 1: \quad H = I - 2uu^T$$

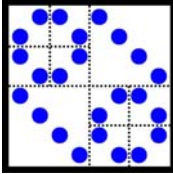
H as rank-1 perturbation of the identity is symmetric, idempotent and orthogonal:

$$H^T = I - 2uu^T = H$$

$$\begin{aligned} H^T H &= H^2 = (I - 2uu^T)(I - 2uu^T) = \\ &= I - 2uu^T - 2uu^T + 4\underbrace{uu^T uu^T}_1 = I \end{aligned}$$

For complex problems:

orthogonal \rightarrow unitary, symmetric \rightarrow hermitian



Use H_1 with appropriate vector u_1 to eliminate first column of A

$$H_1 A = (I - 2u_1 u_1^T)(a_1 \quad \dots \quad a_m) = (a_1 - 2(u_1^T a_1)u_1 \quad \dots \quad *) = \begin{pmatrix} \alpha & * \\ 0 & * \\ \vdots & * \\ 0 & * \end{pmatrix}$$

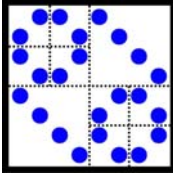
\downarrow
 αe_1

To satisfy this equation we have to find a vector u_1 of length 1 with $a_1 - 2(u_1^T a_1)u_1 = \alpha e_1$

Because H_1 is orthogonal it holds:

$$\|H_1 a_1\|_2 = \|a_1\|_2 = \|\alpha e_1\|_2 = |\alpha| \Rightarrow \alpha = \pm \|a_1\|_2 \quad , \text{ e.g. } \alpha = \|a_1\|_2$$

$$u_1 = \frac{a_1 - \|a_1\|_2 e_1}{2(u_1^T a_1)} = \frac{a_1 - \|a_1\|_2 e_1}{\|a_1 - \|a_1\|_2 e_1\|_2}$$



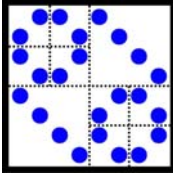
Repeat for all columns of A:

$$H_1 A = H_1 A_1 = (I - 2u_1 u_1^T) A = \left(\begin{array}{c|ccc} \|a_1\|_2 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array} \right)$$

Apply the same procedure on A_2 , $n-1 \times m-1$ matrix.

$$\tilde{H}_2 A_2 = (I - 2\tilde{u}_2 \tilde{u}_2^T) A_2 = \left(\begin{array}{c|ccc} \alpha_2 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & A_3 & \\ 0 & & & \end{array} \right)$$

Extend $u_2 := \begin{pmatrix} 0 \\ \tilde{u}_2 \end{pmatrix}$, $H_2 := I - 2u_2 u_2^T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \tilde{H}_2 & \\ 0 & & & \end{pmatrix}$



For column $1, 2, \dots, m$ this gives Householder matrices H_1, \dots, H_m with

$$\underbrace{H_m \cdots H_2 H_1}_{Q^T} \cdot A = H_m \cdots H_3 \cdot \begin{pmatrix} \alpha_1 & * & * & * \\ 0 & \alpha_2 & * & * \\ 0 & 0 & \boxed{A_3} & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = R$$

Hence: $A = QR$

$$Q := (H_m \cdots H_2 H_1)^T = H_1 H_2 \cdots H_m$$

Remark: For $m=n$:

H_1, \dots, H_{m-1} is enough, because last column is scalar