Erster Schritt Givens-QR-Zerlegung

$$G_{21}A = \left(\begin{array}{cc|ccc} \cos(\phi_{21}) & \sin(\phi_{21}) & & & \\ \sin(\phi_{21}) & -\cos(\phi_{21}) & & & \\ \hline & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{array}\right) \bullet \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nm} \end{pmatrix}$$

$$\rightarrow \left(\begin{array}{ccccc} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1m} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2m} \\ \hline a_{31} & a_{32} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nm} \end{array}\right)$$

$$\phi_{21} := arcctg\left(\frac{a_{11}}{a_{21}}\right)$$

$$a_{1k} \rightarrow \cos(\phi_{21})a_{1k} + \sin(\phi_{21})a_{2k}$$

$$a_{2k} \rightarrow \sin(\phi_{21})a_{1k} - \cos(\phi_{21})a_{2k}$$

Zweiter Schritt

$$G_{31}\left(G_{21}A\right)=\begin{pmatrix} \cos(\phi_{31}) & & \sin(\phi_{31}) & & \\ & 1 & & & \\ \sin(\phi_{31}) & & -\cos(\phi_{31}) & & \\ \hline & & & 1 & \\ & & & & \ddots \end{pmatrix} \bullet \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1m} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nm} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} a''_{11} & a''_{12} & a''_{13} & \cdots & a''_{1m} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2m} \\ 0 & a'_{32} & \cdots & \cdots & a'_{3m} \\ \hline a_{41} & a_{42} & \cdots & \cdots & a_{4m} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nm} \end{pmatrix}$$

(n-1)-ter Schritt

$$G_{n1}\left(G_{n-1,1}\cdots G_{21}A\right)=\begin{pmatrix} \cos(\phi_{n1}) & & & & \sin(\phi_{n1}) \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \sin(\phi_{n1}) & & & & -\cos(\phi_{n1}) \end{pmatrix}\bullet\begin{pmatrix} \boxed{\tilde{a}_{11}} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \boxed{\tilde{a}_{1m}} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a'_{n-1,2} & a'_{n-1,3} & \cdots & a'_{n-1,m} \\ \boxed{a_{n1}} & a_{n2} & a_{n3} & \cdots & \boxed{a_{nm}} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2m} \\ 0 & a'_{32} & \cdots & \cdots & a'_{3m} \\ \vdots & \vdots & & & \vdots \\ 0 & a'_{n-1,2} & \cdots & \cdots & a'_{n-1,m} \\ 0 & a'_{n2} & \cdots & \cdots & a'_{nm} \end{pmatrix}$$

1. Fall: m>n, voller Rang, n Zeilen, m Spalten

$$\left(\begin{array}{ccccc|ccc} * & * & * & \cdots & & * & \cdots & * \\ G_{21} & * & * & \cdots & & * & \cdots & * \\ G_{31} & G_{32} & * & \cdots & & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & & \vdots & & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{n,n-1} & & * & \cdots & * \end{array}\right) \Rightarrow \left(R \;\middle|\; *\right)$$

$$(n-1)+(n-2)+\cdots 1 = n(n-1)/2$$ Givensrotationen

$$6\big((n-1)(m-1)+(n-2)(m-2)+1(m-n+1)\big) = 6\sum_{j=1}^{n-1} j(m-n+j) = 3mn^2 - n^3 + \ldots$$ Kosten

2. Fall: m<=n, voller Rang

$$\left(\begin{array}{ccccc} * & \cdots & * & \cdots & * \\ G_{21} & * & * & \cdots & * \\ G_{31} & G_{32} & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ G_{m1} & G_{m2} & \cdots & G_{m,m-1} & * \\ \hline G_{m+1,1} & G_{m+1,2} & \cdots & G_{m+1,m-1} & G_{m+1,m} \\ \vdots & \vdots & & \vdots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{n,m-1} & G_{nm} \end{array}\right) \Rightarrow \left(\begin{array}{c} R \\ \hline 0 \end{array}\right)$$

$$(n-1)+(n-2)+\cdots+(n-m) = mn - m^2/2 + \ldots$$ Givensrotationen

$$6\big((n-1)(m-1)+(n-2)(m-2)+(n-m+1)1\big) = 6\sum_{j=1}^{m-1}(n-m+j)\,j = 3m^2n - m^3 + \ldots$$ Kosten

Grenzfall: n=m

$(n-1)+(n-2)+\cdots+1=n^2/2+\ldots$ Givensrotationen

$$6\left((n-1)^2+(n-2)^2+1^2\right)=6\sum_{j=1}^{n-1} j^2=2n^3+\ldots$$ Kosten

Für n x m – Matrix mit allgemeinen Rang:

Q A P = R mit Permutation P (Spaltenvertauschung):

$$AP=\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}P=\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{pmatrix} \qquad R=\begin{pmatrix}R_1 & *\\ 0 & 0\end{pmatrix}$$