High Performance Computing – Programming Paradigms and Scalability

Part 6: Examples of Parallel Algorithms

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Overview

- matrix operations
- JACOBI and GAUSS-SEIDEL iterations
- sorting

Everything that can be invented has been invented.
—Charles H. Duell
commissioner U.S. Office of Patents, 1899

Matrix Operations

- reminder: matrix
  - underlying basis for many scientific problems is a matrix
  - stored as 2-dimensional array of numbers (integer, float, double)
    - row-wise in memory (typical case)
    - column-wise in memory
  - typical matrix operations (K: set of numbers)

    1) $A + B = C$ with A, B, and $C \in K^{N \times M}$

    2) $A \cdot b = c$ with $A \in K^{N \times M}$, $b \in K^{M}$, and $c \in K^{N}$

    3) $A \cdot B = C$ with $A \in K^{N \times M}$, $B \in K^{M \times L}$, and $C \in K^{N \times L}$
  - matrix-vector multiplication (2) and matrix multiplication (3) are main building blocks of numerical algorithms
  - both easy to implement in sequential ➔ what happens in parallel?

Matrix Operations

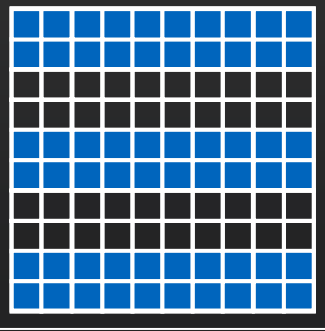
- matrix-vector multiplication
  - appearances
    - systems of linear equations (SLE) $A \cdot x = b$
    - iterative methods for solving SLEs (conjugate gradient, e.g.)
    - implementation of neural networks (determination of output values, training neural networks)
  - standard sequential algorithm for $A \in K^{N \times N}$ and $b, c \in K^N$

```
for i ← 1 to N do
    c[i] ← 0;
    for j ← 1 to N do
        c[i] ← c[i] + A[i][j]*b[j];
    od
od
```

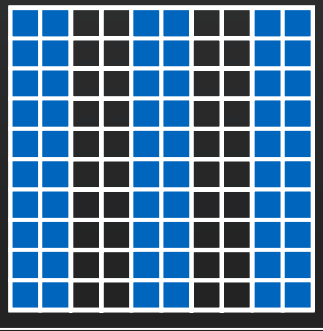
  - for full matrix A this algorithm has a complexity of $O(N^2)$

Matrix Operations

- matrix-vector multiplication (cont'd)
  - in parallel exist three main options to distribute data among *P* procs
    - *row-wise block-striped decomposition*: each process is responsible for a contiguous part of about *N*/*P* rows of *A*
    - *column-wise block-striped decomposition*: each process is responsible for a contiguous part of about *N*/*P* columns of *A*
    - *checkerboard block decomposition*: each process is responsible for a contiguous block of matrix elements
  - vector *b* may be either replicated or block-decomposed itself

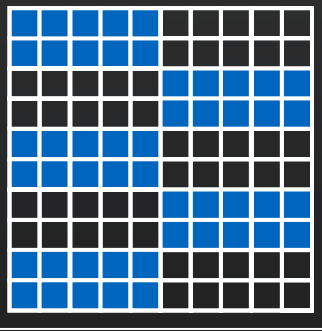


row-wise



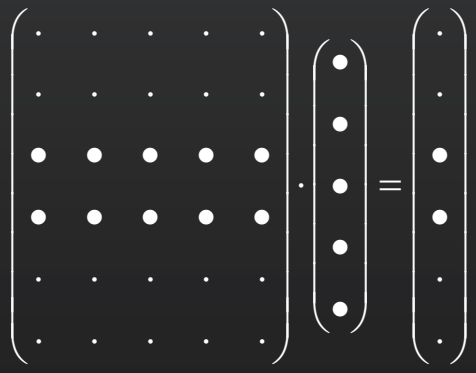
column-wise

checkerboard



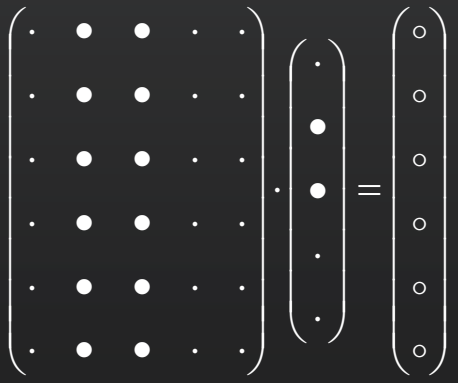
Matrix Operations

- matrix-vector multiplication (cont'd)
  - row-wise block-striped decomposition
    - probably the most straightforward approach
      - each process gets some rows of *A* and entire vector *b*
      - each process computes some components of vector *c*
      - build and replicate entire vector *c* (gather-to-all, e.g.)
    - complexity of $O(N^2/P)$ multiplications / additions for *P* processes



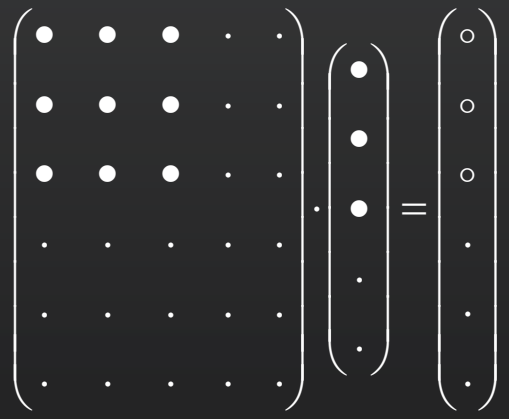
Matrix Operations

- matrix-vector multiplication (cont'd)
  - column-wise block-striped decomposition
    - less straightforward approach
      - each process gets some columns of *A* and respective elements of vector *b*
      - each process computes partial results of vector *c*
      - build and replicate entire vector *c* (all-reduce or maybe a reduce-scatter if processes do not need entire vector *c*)
    - complexity is comparable to row-wise approach



Matrix Operations

- matrix-vector multiplication (cont'd)
  - checkerboard block decomposition
    - each process gets some block of elements of *A* and respective elements of vector *b*
    - each process computes some partial results of vector *c*
    - build and replicate entire vector *c* (all-reduce, but "unused" elements of vector *c* have to be initialised with zero)
    - complexity of the same order as before; it can be shown that checkerboard approach has slightly better scalability properties (increasing *P* does not require to increase *N*, too)



Matrix Operations

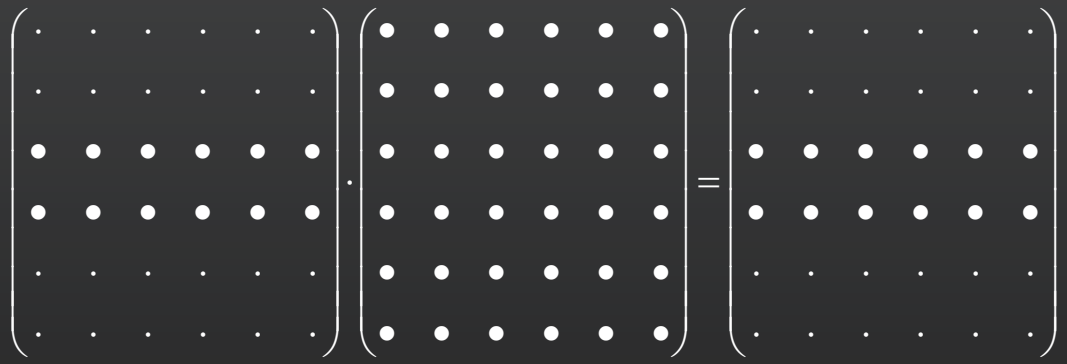
- matrix multiplication
  - appearances
    - computational chemistry (computing changes of state, e.g.)
    - signal processing (DFT, e.g.)
  - standard sequential algorithm for $A, B, C \in K^{N \times N}$

```
for i ← 1 to N do
    for j ← 1 to N do
        C[i][j] ← 0
        for k ← 1 to N do
            C[i][j] ← C[i][j] + A[i][k]*B[k][j];
        od
    od
od
```

  - for full matrices A and B this algorithm has a complexity of $O(N^3)$

Matrix Operations

- matrix multiplication (cont'd)
  - naïve parallelisation
    - each process gets some rows of *A* and entire matrix *B*
    - each process computes some rows of *C*



  - problem: once *N* reaches a certain size, matrix *B* won't fit completely into cache and / or memory ➔ performance will dramatically decrease
  - remedy: subdivision of matrix *B* instead of whole matrix *B*

Matrix Operations

- matrix multiplication (cont'd)
  - recursive algorithm
    - algorithm follows the divide-and-conquer principle
    - subdivide both matrices *A* and *B* into four smaller submatrices

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \qquad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$$

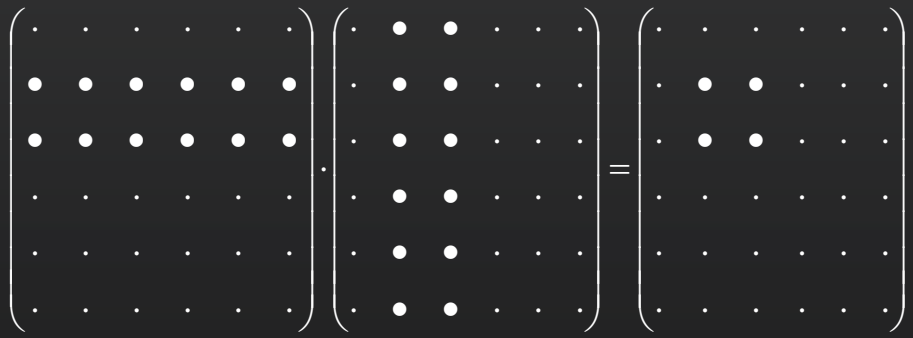
    - hence, the matrix multiplication can be computed as follows

$$C = \begin{pmatrix} A_{00} \cdot B_{00} + A_{01} \cdot B_{10} & A_{00} \cdot B_{01} + A_{01} \cdot B_{11} \\ A_{10} \cdot B_{00} + A_{11} \cdot B_{10} & A_{10} \cdot B_{01} + A_{11} \cdot B_{11} \end{pmatrix}$$

    - if blocks are still too large for the cache, repeat this step (i.e. recursively subdivide) until it fits
    - furthermore, this method has significant potential for parallelisation (especially for MemMS)

Matrix Operations

- matrix multiplication (cont'd)
  - CANNON's algorithm
    - each process gets some rows of *A* and some columns of *B*
    - each process computes some components of matrix *C*
    - different possibilities for assembling the result
      - gather all data, build and (maybe) replicate matrix *C*
      - "pump" data onward to next process (➔ systolic array)
    - complexity of $O(N^3/P)$ multiplications / additions for *P* processes



Overview

- matrix operations
- JACOBI and GAUSS-SEIDEL iterations
- sorting

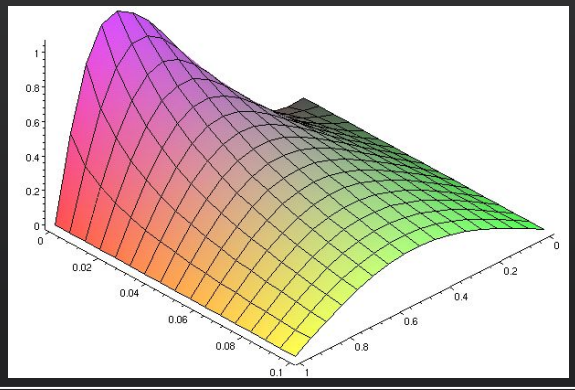
JACOBI and GAUSS-SEIDEL Iterations

- scenario
  - solve an elliptic partial differential equation (PDE) with DIRICHLET boundary conditions on a given domain Ω
  - simple example: POISSON equation $-\Delta u = f$

(1) $$-\Delta u(x,y) = -\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \quad \text{for } (x,y) \in \Omega$$

on the unit square $\Omega =]0,1[^2$ with u given on the boundary of Ω

  - task: $u(x,y)$ or an approximation to it has to be found
  - occurrences of such PDEs
    - fixed membrane
    - stationary heat equation (picture)
    - electrostatic fields
    - …



JACOBI and GAUSS-SEIDEL Iterations

- discretisation
  - for solving our example PDE, a discretisation is necessary
  - typical discretisation techniques
    - finite differences
    - finite volumes
    - finite elements
  - finite difference discretisation (forward and backward differences) for the second derivatives in (1) for a mesh width *h* leads to

$$\frac{\partial^2 u(x,y)}{\partial x^2} \approx \frac{u(x-h,y)-2u(x,y)+u(x+h,y)}{h^2}$$

(2)

$$\frac{\partial^2 u(x,y)}{\partial y^2} \approx \frac{u(x,y-h)-2u(x,y)+u(x,y+h)}{h^2}$$

JACOBI and GAUSS-SEIDEL Iterations

- discretisation (cont'd)
  - for computational solution of our problem a 2D grid is necessary
    - equidistant grid with $(N+1)\times(N+1)$ grid points, $N = 1/h$
    - $u_{i,j} \approx u(i\cdot h, j\cdot h)$ with $i, j = 0, \ldots, N$
  - hence, (2) can be written as

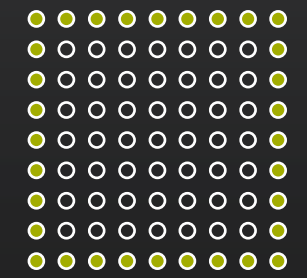
(3) $$\frac{\partial^2 u(x,y)}{\partial x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \qquad \frac{\partial^2 u(x,y)}{\partial y^2} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$

  - with (3) and an appropriate discretisation of $f(x,y)$ follows

(4) $$-u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1} = h^2 \cdot f_{i,j} \qquad 0 < i, j < N$$

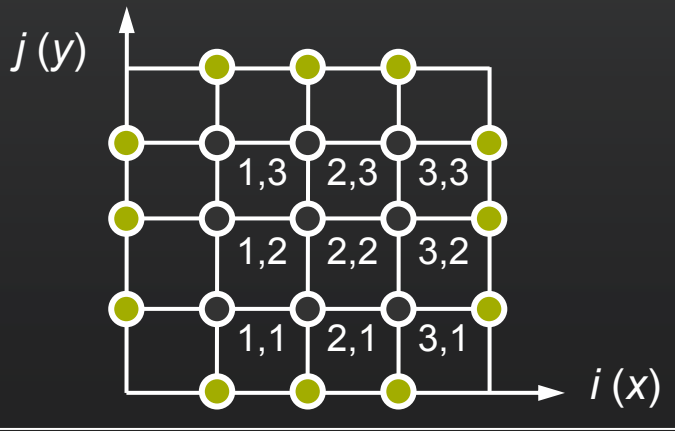
  - resulting equation on the boundary

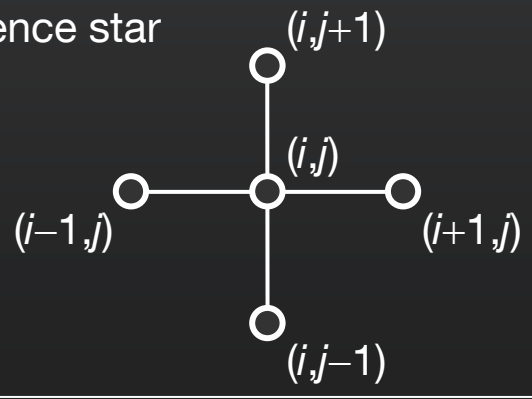
$$u_{i,j} = g_{i,j} \qquad i, j = 0 \text{ or } i, j = N$$



JACOBI and GAUSS-SEIDEL Iterations

- system of linear equations
  - for each inner point there is one linear equation according to (4)
  - equations in points next to the boundary (i.e. $i, j = 1$ or $i, j = N-1$) access boundary values $g_{i,j}$
    - these are shifted to the right-hand side of the equation
    - hence, all unknowns are located left, all known quantities right
  - assembling of the overall vector of unknowns by lexicographic row-wise ordering





JACOBI and GAUSS-SEIDEL Iterations

- system of linear equations (cont'd)
  - this results to a system of linear equations $A \cdot x = b$
    - with $(N-1)^2$ equations in $(N-1)^2$ unknowns
    - matrix A has block-tridiagonal structure and is sparse
- (5)

$$\underbrace{\begin{pmatrix} 4 & -1 & & & -1 & & & & \\ -1 & 4 & -1 & & & \ddots & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & \\ & & \ddots & \ddots & \ddots & & & \ddots & \\ & & & \ddots & \ddots & \ddots & & & -1 \\ -1 & & & & \ddots & \ddots & \ddots & & \\ & \ddots & & & & \ddots & \ddots & \ddots & \\ & & \ddots & & & & \ddots & \ddots & \ddots \\ & & & \ddots & & & -1 & 4 & -1 \\ & & & & -1 & & & -1 & 4 \end{pmatrix}}_{=A} \cdot \underbrace{\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{N-1,1} \\ u_{1,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_{N-1,N-1} \end{pmatrix}}_{=x} = \underbrace{\begin{pmatrix} f_{1,1} \\ \vdots \\ f_{N-1,1} \\ f_{1,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{N-1,N-1} \end{pmatrix}}_{=b}$$

JACOBI and GAUSS-SEIDEL Iterations

- solving large sparse SLEs
  - schoolbook method for solving SLEs: GAUSSian elimination
    - direct solver that provides the exact solution
    - has a complexity of $O(M^3)$ for M unknowns (!)
    - does not exploit sparsity of matrix A that is even filled-up
  - hence, using some iterative method instead
    - approximates the exact solution
    - has a complexity of $O(M)$ operations for a single iteration
    - typically much less than $O(M^2)$ iteration steps needed ➔ ideal case of $O(1)$ steps for *multigrid* or *multilevel* methods
    - basic methods (number of steps depending on M)
      - relaxation methods: JACOBI, GAUSS-SEIDEL, SOR
      - minimisation methods: steepest descent, CG

JACOBI and GAUSS-SEIDEL Iterations

- JACOBI iteration
  - decompose matrix A in its diagonal part D_A, its upper triangular part U_A, and its lower triangular part L_A

$$A = L_A + D_A + U_A$$

  - starting with

$$b = A \cdot x = D_A \cdot x + (L_A + U_A) \cdot x$$

    and writing $b = D_A \cdot x^{(T+1)} + (L_A + U_A) \cdot x^{(T)}$ with $x^{(T)}$ denoting the approximation to x after T steps of the iteration leads to the following iterative scheme

$$x^{(T+1)} = -D_A^{-1} \cdot (L_A + U_A) \cdot x^{(T)} + D_A^{-1} \cdot b = x^{(T)} + D_A^{-1} \cdot r^{(T)}$$

    where the residual is defined as $r^{(T)} = b - A \cdot x^{(T)}$

JACOBI and GAUSS-SEIDEL Iterations

- JACOBI iteration (cont'd)
  - algorithmic form of the JACOBI iteration

```
for T ← 0, 1, 2, ... do
    for k ← 1 to M do
        x_k^(T+1) ← 1/A_k,k * (b_k − Σ_{j≠k} A_k,j * x_j^(T))
    od
od
```

  - for our example with matrix *A* according to (5) this means

```
for T ← 0, 1, 2, ... do
    for j ← 1 to N−1 do
        for i ← 1 to N−1 do
            u_i,j^(T+1) ← ¼ * (u_i−1,j^(T) + u_i,j−1^(T) + u_i+1,j^(T) + u_i,j+1^(T) − h^2 * f_i,j)
        od
    od
od
```

JACOBI and GAUSS-SEIDEL Iterations

- GAUSS-SEIDEL iteration
  - same decomposition of matrix A as for JACOBI

$$A = L_A + D_A + U_A$$

  - starting with

$$b = A \cdot x = (D_A + L_A) \cdot x + U_A \cdot x$$

  and writing $b = (D_A + L_A) \cdot x^{(T+1)} + U_A \cdot x^{(T)}$ leads to the following iterative scheme

$$x^{(T+1)} = -(D_A + L_A)^{-1} \cdot U_A \cdot x^{(T)} + (D_A + L_A)^{-1} \cdot b = x^{(T)} + (D_A + L_A)^{-1} \cdot r^{(T)}$$

  where the residual is defined as $r^{(T)} = b - A \cdot x^{(T)}$

JACOBI and GAUSS-SEIDEL Iterations

- GAUSS-SEIDEL iteration (cont'd)
  - algorithmic form of the GAUSS-SEIDEL iteration

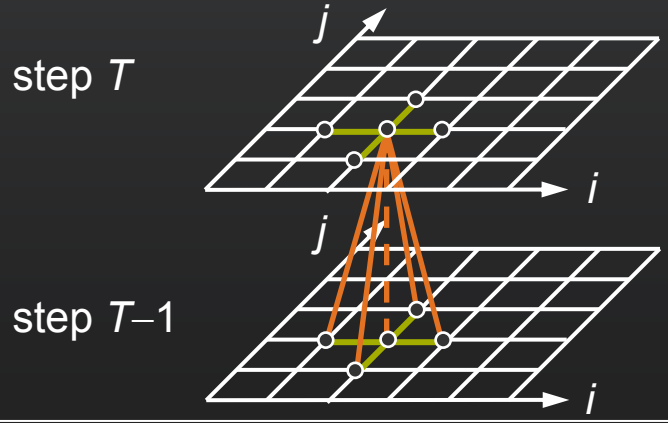
```
for T ← 0, 1, 2, ... do
    for k ← 1 to M do
        x_k^(T+1) ← 1/A_k,k*(b_k − Σ_{j<k} A_k,j*x_j^(T+1) − Σ_{j>k} A_k,j*x_j^(T))
    od
od
```

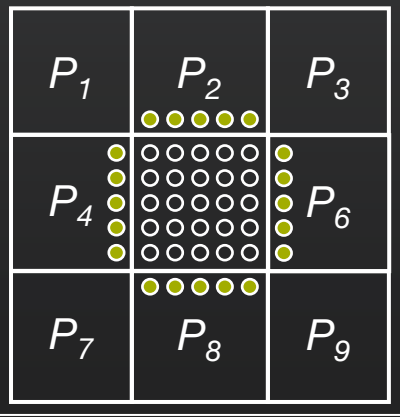
  - for our example with matrix *A* according to (5) this means

```
for T ← 0, 1, 2, ... do
    for j ← 1 to N−1 do
        for i ← 1 to N−1 do
            u_i,j^(T+1) ← ¼*(u_i−1,j^(T+1) + u_i,j−1^(T+1) + u_i+1,j^(T) + u_i,j+1^(T) − h²*f_i,j)
        od
    od
od
```

JACOBI and GAUSS-SEIDEL Iterations

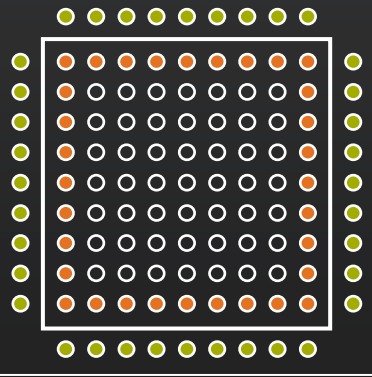
- parallelisation of JACOBI iteration
  - neither JACOBI nor GAUSS-SEIDEL are used today very frequently for solving large SLEs (they are too slow)
  - nevertheless, the algorithmic aspects are still of interest
  - a parallel JACOBI is quite straightforward
    - in iteration step T only values from step $T-1$ are used
    - hence, all updates of one step can be made in parallel
    - furthermore, subdivide the domain into strips or blocks, e.g.





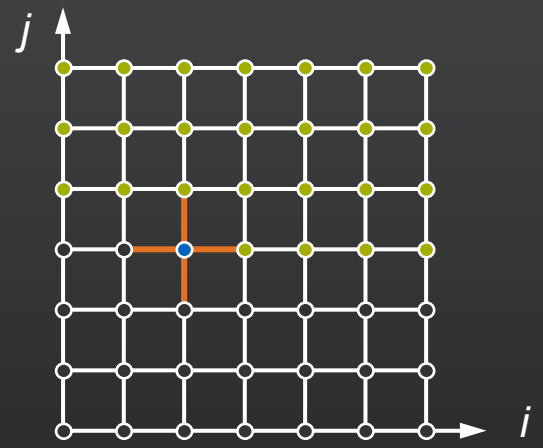
JACOBI and GAUSS-SEIDEL Iterations

- parallelisation of JACOBI iteration (cont'd)
  - for its computations, each processor *P* needs
    - a subset of boundary values (if *P* is adjacent to the boundary)
    - one row / column of values from *P*'s neighbouring processes
    - a global / local termination condition
  - hence, each processor has to execute the following algorithm
    1) update all local approximate values $u_{i,j}^{(T)}$ to $u_{i,j}^{(T+1)}$
    2) send all updates (•) to the respective neighbouring processes
    3) receive all necessary updates (•) from neighbouring processes
    4) compute local residual values and perform a reduce-all for global residual
    5) continue if global residual $> \varepsilon$



JACOBI and GAUSS-SEIDEL Iterations

- parallelisation of GAUSS-SEIDEL iteration
  - problem: since the updated values are immediately used where available, parallelisation seems to be quite complicated

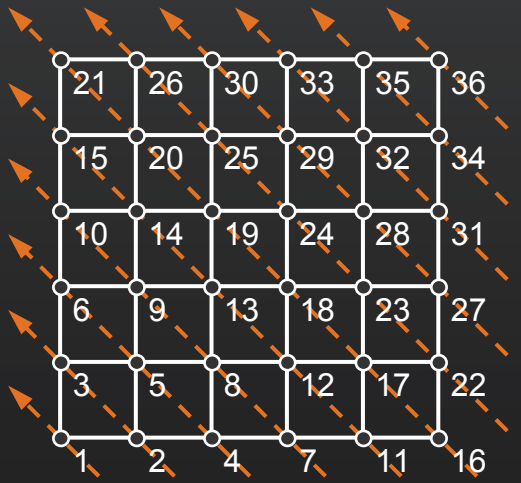


- updated values (step T)
- old values (step $T-1$)

  - hence, a different order of visiting / updating the grid points is necessary
    - wavefront ordering
    - red-black or checkerboard ordering

JACOBI and GAUSS-SEIDEL Iterations

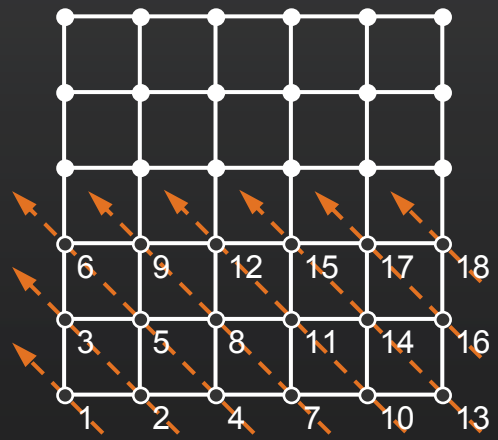
- parallelisation of GAUSS-SEIDEL iteration (cont'd)
  - wavefront ordering (1)
    - diagonal ordering of updates ➔ all values along a diagonal line can be updated in parallel; single diagonal lines still have to be processed sequentially
    - problem: for $P = N$ processors there are P^2 updates that need $2P-1$ sequential steps ➔ speed-up restricted to $P/2$



P_1:	1	2	4	7	11	16	22	27	31	34	36
P_2:		3	5	8	12	17	23	28	32	35	
P_3:			6	9	13	18	24	29	33		
P_4:				10	14	19	25	30			
P_5:					15	20	26				
P_6:						21					

JACOBI and GAUSS-SEIDEL Iterations

- parallelisation of GAUSS-SEIDEL iteration (cont'd)
  - wavefront ordering (2)
    - better
      - row-wise decomposition of matrix A in K blocks of N/K rows
      - for $P = N/K$ processors there are K sequential blocks of $K \cdot P^2$ updates that need $K \cdot P + P - 1$ sequential steps each
      - hence, speed-up restricted to $K \cdot P/(K+1)$

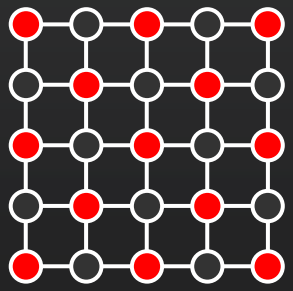


P_1:	1	2	4	7	10	13	16	18
P_2:		3	5	8	11	14	17	
P_3:			6	9	12	15		

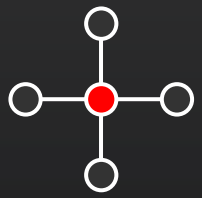
here, $K = 2$ ➔ speed-up $S(p) = 2P/3$

JACOBI and GAUSS-SEIDEL Iterations

- parallelisation of GAUSS-SEIDEL iteration (cont'd)
  - red-black or checkerboard ordering
    - grid points get a checkerboard colouring of red and black
    - lexicographic order of visiting / updating the grid points
      - first the red ones, than the black ones
      - hence, no dependencies within red nor within black set
    - subdivide grid such that each processor gets some red and some black points ➔ two sequential steps necessary, but perfect parallelism within each of them



red-black ordering



5-point star for red (left) and black (right) grid points

Overview

- matrix operations
- JACOBI and GAUSS-SEIDEL iterations
- sorting

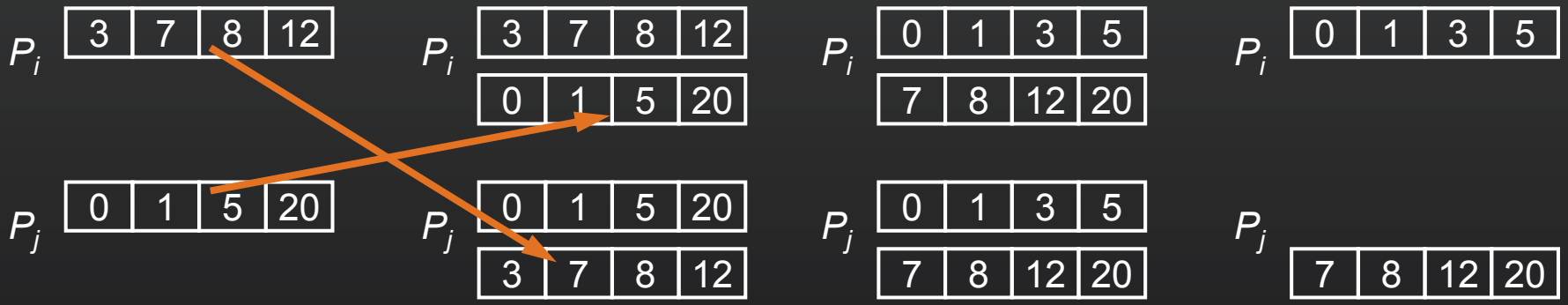
Sorting

- reminder: sorting
  - one of the most common operations performed by computers
  - let $A = \langle a_1, a_2, \ldots, a_N \rangle$ be a sequence of N elements in arbitrary order
  - sorting transforms A into a monotonically increasing or decreasing sequence $\tilde{A} = \langle \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N \rangle$ such that
    - $\tilde{a}_i \leq \tilde{a}_j$ for $1 \leq i \leq j \leq N$ *(increasing order)*
    - $\tilde{a}_i \geq \tilde{a}_j$ for $1 \leq i \leq j \leq N$ *(decreasing order)*

    and $\tilde{A}$ being a permutation of A
  - in general, sorting algorithms are comparison-based, i.e. an algorithm sorts an unordered sequence of elements by repeatedly comparing / exchanging pairs of elements
  - lower bound of the sequential complexity of any comparison-based algorithm is $O(N \cdot \log N)$

Sorting

- basic operations
  - in sequential / parallel sorting algorithms, some basic operations are repeatedly executed
    - *compare-exchange*: elements a_i and a_j are compared and exchanged in case they are out of sequence
    - *compare-split*: already sorted blocks of elements A_i and A_j stored at different processors P_i and P_j, resp., are merged and split in the following manner



P_i sends block A_i to P_j and vice versa — merge — split

Sorting

- bubble sort
  - simple comparison-based sorting algorithm of complexity $O(N^2)$
  - standard sequential algorithm for sorting sequence A

```
for i ← N to 2 by –1 do
    for j ← 1 to i–1 do
        compare-exchange (a[j], a[j+1])
    od
od
```

  - example: iterations $i = 8, 7, 6$ for sorting $A = \langle 8, 7, 6, 5, 4, 3, 2, 1 \rangle$

initial setup	8	7	6	5	4	3	2	1
1st iteration	7	6	5	4	3	2	1	8
2nd iteration	6	5	4	3	2	1	7	8
3rd iteration	5	4	3	2	1	6	7	8

Sorting

- bubble sort (cont'd)
  - standard algorithm not very suitable for parallelisation ➔ partition of *A* into blocks of size *N*/*P* elements (for *P* processors) still to be processed sequentially
  - hence, different approach necessary: *odd-even transposition*
    - *idea*: sorting *N* elements (*N* is even) in *N* phases, each of which requires *N*/2 compare-exchange operations ➔ alternation between two phases, called odd and even phase
    - odd phase: only elements with odd indices are compare-exchanged with right neighbours ➔ (a_1, a_2), (a_3, a_4), (a_5, a_6), …
    - even phase: only elements with even indices are compare-exchanged with right neighbours ➔ (a_2, a_3), (a_4, a_5), (a_6, a_7), …
    - after *N* phases of odd-even-exchanges, sequence *A* is sorted

Sorting

- bubble sort (cont'd)
  - example: odd-even-transp. for sorting $A = \langle 8, 7, 6, 5, 4, 3, 2, 1 \rangle$

Sorting

- bubble sort (cont'd)
  - parallelisation of odd-even-transposition
    - each process is assigned a block of N/P elements, which are sorted internally (using merge sort or quicksort, e.g.) with a complexity of $O((N/P)\cdot\log(N/P))$
    - afterwards, each processor executes P phases ($P/2$ odd and $P/2$ even ones) of compare-split operations
    - at the end, sequence A is sorted (and stored distributed over P processes where process P_i holds block A_i with $A_i \leq A_j$ for $i < j$)
    - during each phase $O(N/P)$ comparisons are performed, thus, the total complexity of the parallel sort can be computed as

$$\underbrace{O((N/P)\cdot\log(N/P))}_{\text{local sort}} + \underbrace{O(N)}_{\text{comparisons}} + \text{communication}$$

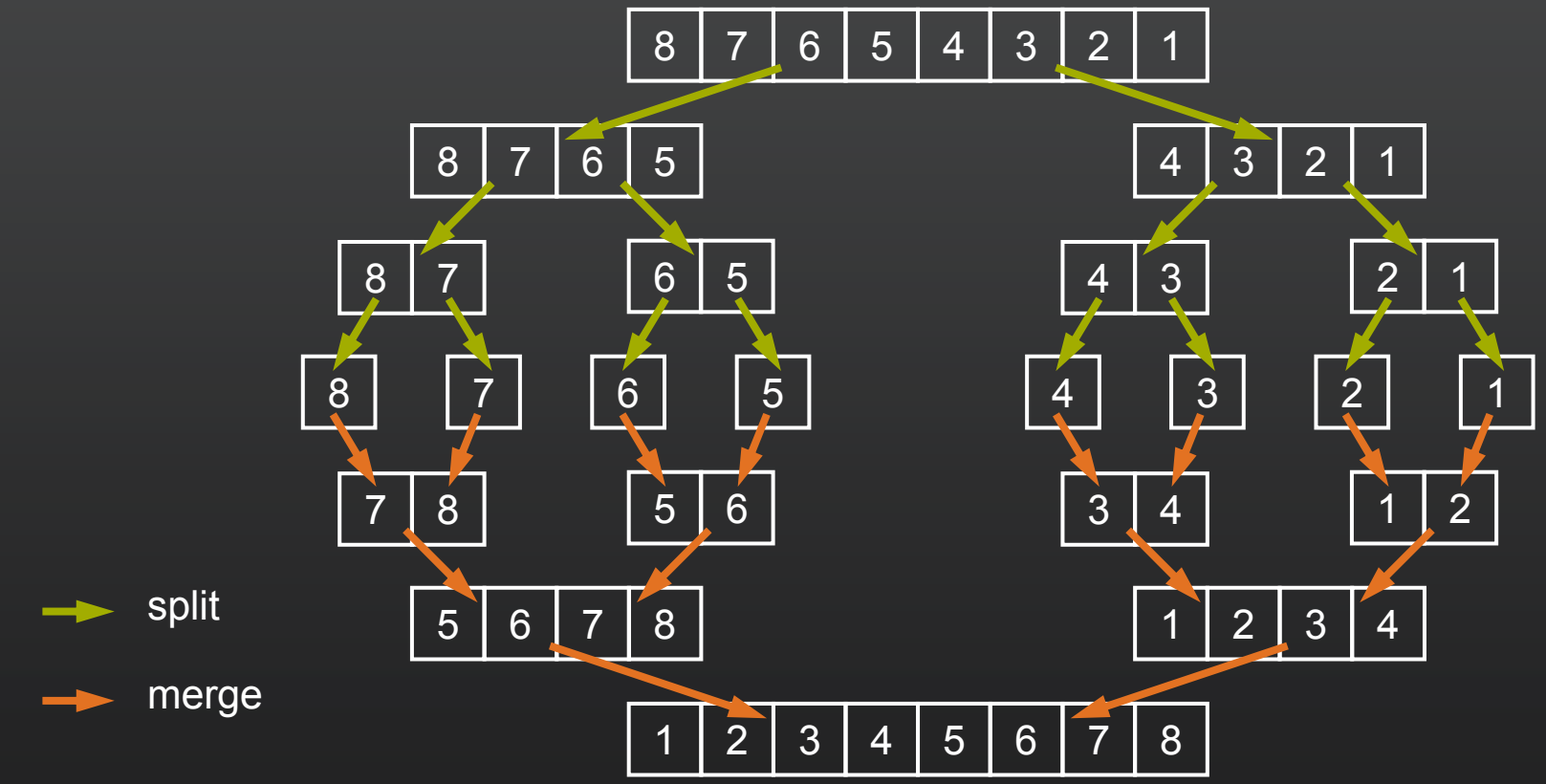
Sorting

- merge sort
  - comparison-based sorting algorithm of complexity $O(N \cdot \log N)$
  - based on the divide-and-conquer (DAC) strategy
  - basic idea: construct a sorted list by merging two sorted lists

```
procedure mergesort (list L)
    if (size of L = 1) return L
    else
        divide L into left and right list
        left ← mergesort (left)
        right ← mergesort (right)
        result ← merge (left, right)
        return result
    fi
end
```

Sorting

- merge sort (cont'd)
  - example: merge sort for sequence $A = \langle 8, 7, 6, 5, 4, 3, 2, 1 \rangle$

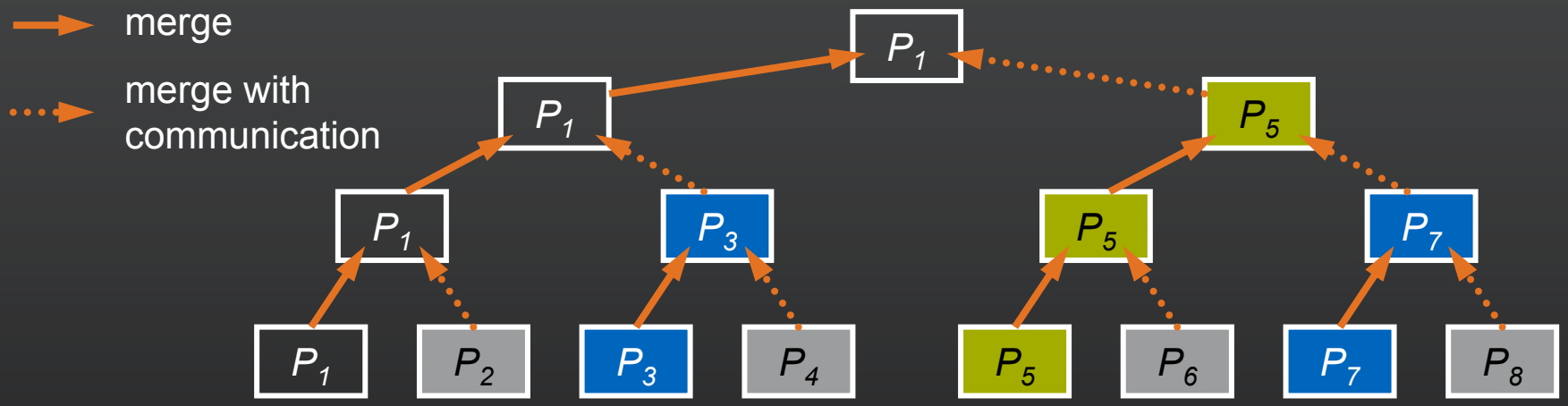


Sorting

- merge sort (cont'd)
  - parallelisation of merge sort: *naïve approach*
    - construct a binary processing tree of P leaf nodes and assign P processors to tree nodes (i.e. inner and leaf nodes)
    - divide sequence A into blocks A_i of size N/P and store them at leaf nodes
    - parallel sort of blocks A_i via sequential merge sort with a complexity of $O((N/P) \cdot \log (N/P))$
    - repeated parallel merge of sorted sublists bottom-up the tree with a total complexity of $O(N)$ (merge operation at the different tree levels consist of $N + N/2 + N/4 + \ldots + N/P$ comparisons)
  - problem: sending of sublists might induce heavy communication
  - hence, appropriate mapping of procs to nodes is indispensable

Sorting

- merge sort (cont'd)
  - example: mapping of processors to tree nodes for $P = 8$



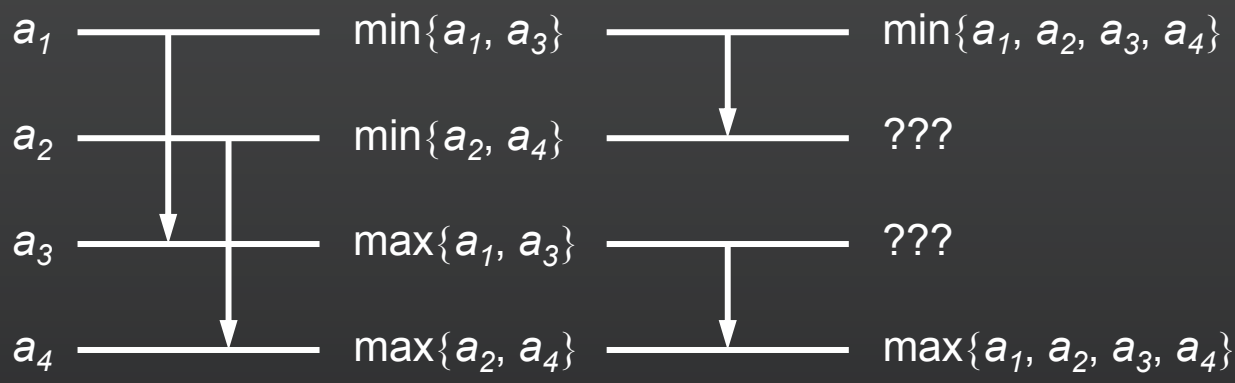
  - observations: not to expect very good speed-up values for parallel merge of sublists (amount of used processors is halved in every step ➔ estimate of MINSKY)
  - hence, different strategy for parallel merge necessary

Sorting

- sorting networks
  - sorting networks are based on a comparison network model, that sort N elements in significantly smaller than $O(N \cdot \log N)$ operations
  - key component of a sorting network: comparator
    - device with two inputs a_1, a_2 and two outputs $\tilde{a}_1$, $\tilde{a}_2$
    - *increasing comparator*: $\tilde{a}_1 = \min\{a_1, a_2\}$ and $\tilde{a}_2 = \max\{a_1, a_2\}$
    - *decreasing comparator*: $\tilde{a}_1 = \max\{a_1, a_2\}$ and $\tilde{a}_2 = \min\{a_1, a_2\}$
  - sorting networks consist of several columns of such comparators, where each column performs a permutation, thus the final column is sorted in increasing / decreasing order

Sorting

- sorting networks (cont'd)
  - first (naïve) idea of a parallel sorter

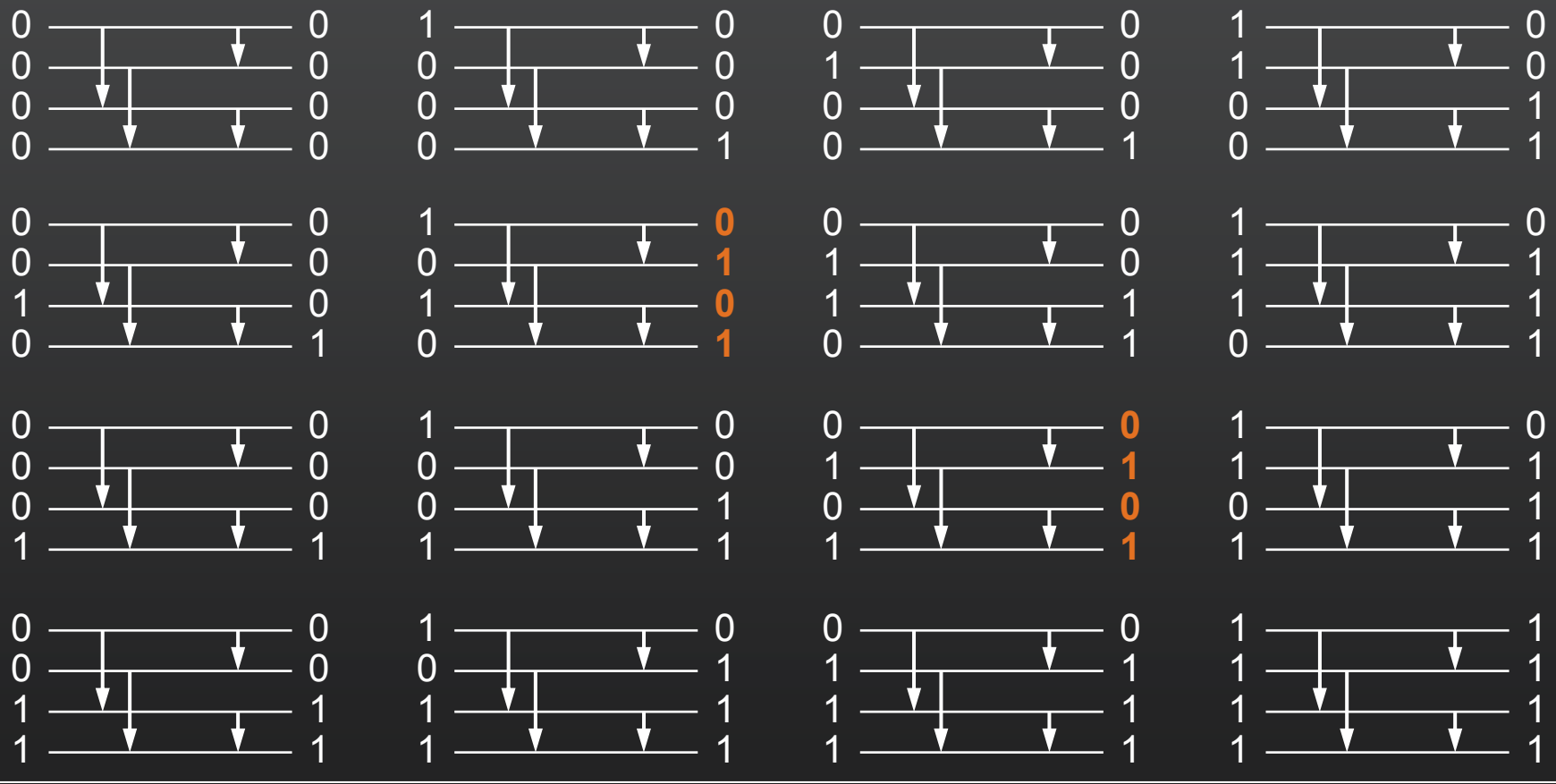


  - weak aspects
    - does not work for all permutations of input sequences
    - question: which permutations lead to incorrect results (N! tests)

Any network sorts correct any input sequence iff it sorts all possible 0-1-bit vectors ➔ only 2^N tests necessary (but for large N still impossible)

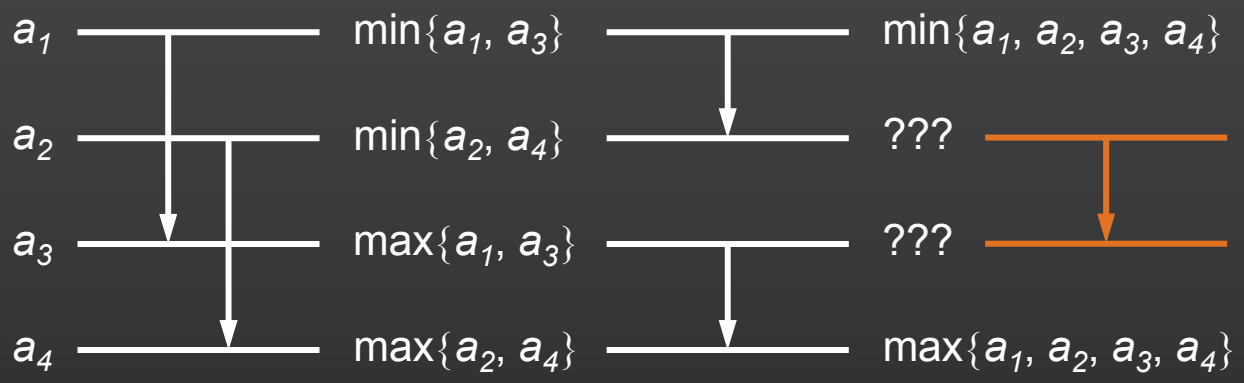
Sorting

- sorting networks (cont'd)
  - all possible 0-1-bit vectors for naïve parallel sorter



Sorting

- sorting networks (cont'd)
  - modified (naïve) parallel sorter



  - corresponding 0-1-bit vectors

  - for N inputs around $O(N)$ steps necessary ➔ could this be done better and how to construct an efficient parallel sorter?

Sorting

- bitonic sort
  - a bitonic sorting network sorts N elements in $O(\log^2 N)$ operations
  - key task: rearrangement of a bitonic sequence into a sorted one
  - definition: bitonic sequence

    A sequence $S = \langle a_0, a_1, \ldots, a_{N-1} \rangle$ is bitonic iff

    1) there exists an index i, $0 \leq i \leq N-1$, such that $\langle a_0, \ldots, a_i \rangle$ is monotonically increasing and $\langle a_{i+1}, \ldots, a_{N-1} \rangle$ is monotonically decreasing, or
    2) there exists a cyclic shift of indices so that (1) is satisfied.

  - example
    - $\langle 1, 2, 4, 7, 6, 0 \rangle$ first increases and then decreases
    - $\langle 8, 9, 2, 1, 0, 4 \rangle$ can be cyclic shifted to $\langle 0, 4, 8, 9, 2, 1 \rangle$

Sorting

- bitonic sort (cont'd)
  - let $S = \langle a_0, a_1, \ldots, a_{N-1} \rangle$ be a bitonic sequence such that

$$a_0 \leq a_1 \leq \ldots \leq a_K \quad \text{and} \quad a_K \geq a_{K+1} \geq \ldots \geq a_{N-1}$$

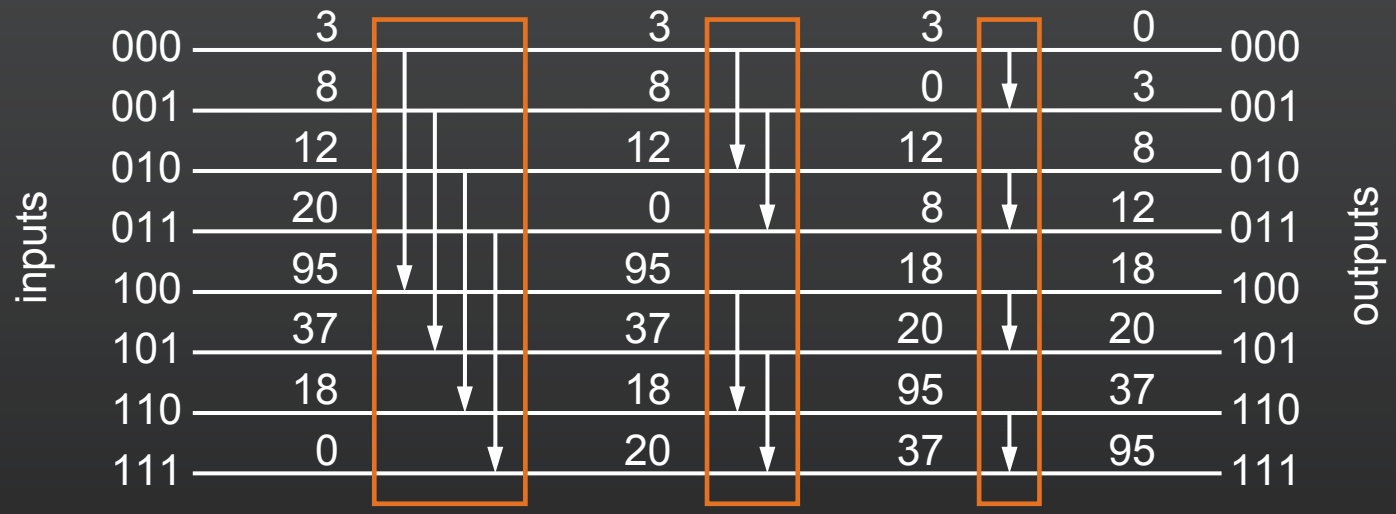
  - consider the following subsequences of S
    - $S_1 = \langle \min\{a_0, a_{N/2}\}, \min\{a_1, a_{N/2+1}\}, \ldots, \min\{a_{N/2-1}, a_{N-1}\} \rangle$
    - $S_2 = \langle \max\{a_0, a_{N/2}\}, \max\{a_1, a_{N/2+1}\}, \ldots, \max\{a_{N/2-1}, a_{N-1}\} \rangle$
  - in sequence S_1, there is an element $s_i = \min\{a_i, a_{N/2+i}\}$ such that all elements before s_i are from the increasing part of S and all elements after s_i are from the decreasing part of S
  - also, in sequence S_2, there is an element $\hat{s}_i = \max\{a_i, a_{N/2+i}\}$ such that all elements before $\hat{s}_i$ are from the decreasing part of S and all elements after $\hat{s}_i$ are from the increasing part of S
  - hence, sequences S_1 and S_2 are bitonic sequences

Sorting

- bitonic sort (cont'd)
  - furthermore, $S_1 \leq S_2$ as s_i is greater than or equal to all elements of S_1, $\hat{s}_i$ is less than or equal to all elements of S_2, and $\hat{s}_i \geq s_i$
  - hence, the initial problem of rearranging a bitonic sequence of size N was reduced to that of rearranging two smaller bitonic sequences of size $N/2$ and concatenating the results
  - this operation is further referred to as *bitonic split*
  - the recursive usage of the bitonic split operation until all obtained subsequences are of size one leads to a sorted output in increasing order ➔ sorting a bitonic sequence using bitonic splits is called *bitonic merge*

Sorting

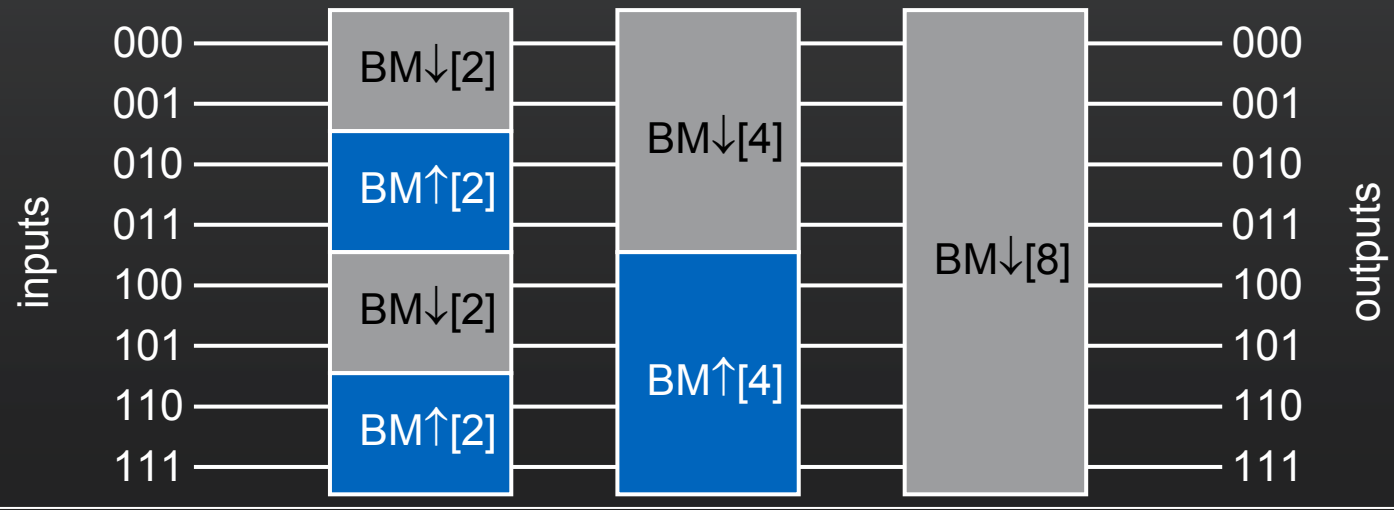
- bitonic sort (cont'd)
  - example: bitonic merging network with 8 inputs (BM↓[8])



initial sequence	3	8	12	20	95	37	18	0
1st bitonic split	3	8	12	0	95	37	18	20
2nd bitonic split	3	0	12	8	18	20	95	37
3rd bitonic split	0	3	8	12	18	20	37	95

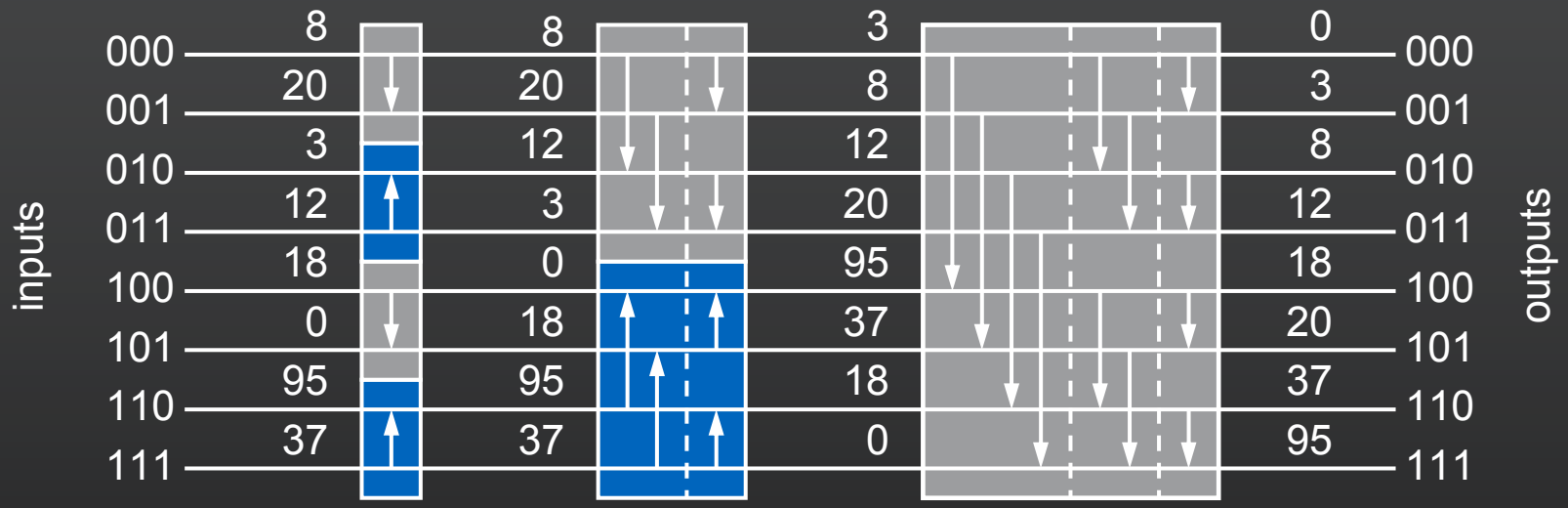
Sorting

- bitonic sort (cont'd)
  - for sorting bitonic sequence in decreasing order, ↓ comparators have to be replaced by ↑ comparators ➔ BM↑[8]
  - problem: how to get a bitonic sequence $S = \langle a_0, a_1, \ldots, a_{N-1} \rangle$ out of N unordered elements
  - construct S by repeatedly merging bitonic sequences of increasing length (here, the last bitonic merge (BM↓[8]) sorts the input)



Sorting

- bitonic sort (cont'd)
  - example: bitonic sorting network with 8 inputs



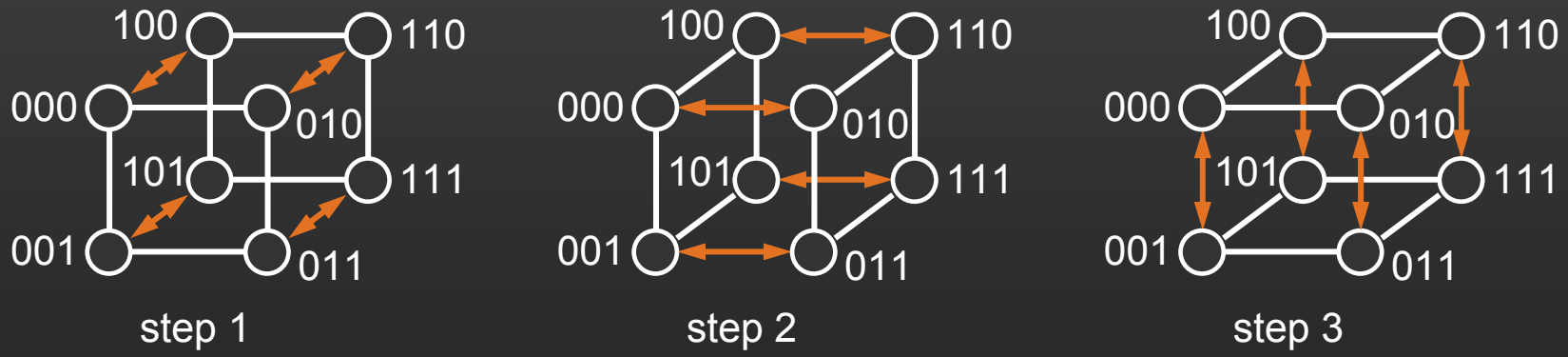
  - depth d(N) of a bitonic sorting network with N inputs can be computed by the following recursion

$$d(N) = d(N/2) + \log N$$

  - hence, $d(N) = \sum_{i=1}^{\log N} i = (\log^2 N + \log N) / 2 = O(\log^2 N)$

Sorting

- bitonic sort (cont'd)
  - the bitonic algorithm is communication intensive ➔ a proper mapping must take into account the underlying network topology
    - *hypercube*: compare-exchange operations take only place between nodes whose labels differ in the K^{th} bit for step K



    - *mesh*: there exist several possibilities for a proper mapping

Sorting

- again: merge sort
  - here, different approach for parallelisation of merge sort using a sorting network instead of a binary tree
  - *idea*
    - divide sequence A into blocks A_i of size N/P and sort them all sequentially (complexity $O((N/P) \cdot \log (N/P))$) in parallel
    - parallel merge
      1) starting point: P sorted sublists, each distributed over one processor
      2) merging two (neighbouring) sublists using compare-split operations leads to $P/2$ sorted sublists, each distributed over two processors
      3) repeatedly executing of step 2 finally leads to one sorted list distributed over P processors

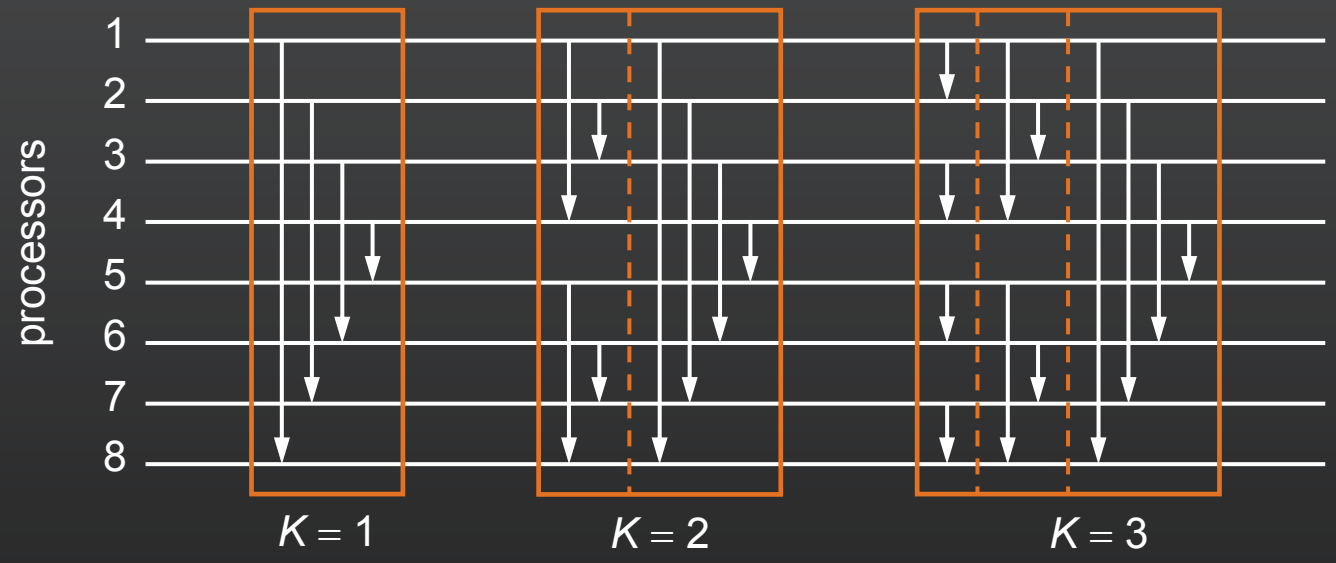
Sorting

- again: merge sort (cont'd)
  - example: repeated merge of sorted sublists for $P = 8$

  - there is a total of log P steps, where in the K^{th} step each processor performs K compare-split operations with its neighbours to obtain (parts of) a sorted list distributed over 2^K nodes
  - hence, the parallel merge can be implemented by a sorting network with a complexity of $O(\log^2 P)$ compare-split operations

Sorting

- again: merge sort (cont'd)
  - example: parallel merge for $P = 8$ processors with $K = 3$ steps



  - total complexity of the parallel merge sort can be computed as

$$\underbrace{O((N/P)\cdot\log(N/P))}_{\text{local sort}} + \underbrace{O((N/P)\cdot\log^2 P)}_{\text{comparisons}} + \text{communication}$$