Solutions

1.

(1) Sequentially: $(ax + by)$ 3 steps, $(bx + cy)$ 3 steps, $(x * (ax + by) + y * (bx + cy))$ 3 steps, total 9 steps sequentially

Parallel: ax, by, bx, cy 1 step in parallel, $(ax + by)$, $(bx + cy)$ 1 step, $x(ax+by)$, $y(bx+cy)$ 1 step, $(x(ax+by)+y(bx+cy))$ 1 step, in total parallel 4 steps.

(2) $ax^2 - 2axy + ay^2 = a(x - y)^2$ 3 steps sequentially and also in parallel

(3) $8x^2 + 22xy + 15y^2 = (2x + 3y)(4x + 5y)$, therefore 7 steps sequentially and 3 steps in parallel.

2.

(1) Pattern of first column of M: $M_1 = (a, b, 0)^T$. Least squares problem:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} * \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} a \\ b \end{pmatrix} = N * \tilde{M}_1 \qquad (1)$$

(2) Normal equations $N^T N \tilde{M}_1 = N^T e_1$,

$$\begin{pmatrix} 5 & -4 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (2)$$

with solution $a = 4/7$ and $b = 3/14$.

3.

(1) $p(x); = (x - \lambda_1)(x - \lambda_2)/(\lambda_1 \lambda_2)$, because then $p(\lambda_i) = 0$, $p(0) = 1$, and p is of degree 2. Then any vector x can be written as $x = ax_1 + bx_2$ in the ONB to A with x_i eigenvector to eigenvalue λ_i, and therefore $p(A)x = ap(A)x_1 + bp(A)x_2 = ap(\lambda_1)x_1 + bp(\lambda_2)x_2 = 0$ for all x, and hence $p(A) = 0$.

$p(x) := (x - \lambda_1)(x - \lambda_2)/(\lambda_1 \lambda_2) = x^2/(\lambda_1 - \lambda_2) - (\lambda_1 + \lambda_2)/(\lambda_1 \lambda_2) + 1$

(2) $0 = p(A) = I - q(A)A = I - (a + bA)A = I - aA - bA^2$ and therefore in comparison with p we can choose for $q(x) = a + bx$: $a = (\lambda_1 + \lambda * 2)/(\lambda_1 - \lambda_2)$ and $b = -1/(\lambda_1 \lambda_2)$.

4.

Undirected graph G:

```
1                   6
| \               / |
|   \           /   |
|     \       /     |
|       \   /       |
2---------4---------7
|         |         |
|         |         |
|         |         |
|         |         |
3---------5---------8
```

(1) Original sparsity pattern: Diagonal entries and subdiagonal entries accordingly. Upper diagonal pattern: (1,2), (1,4), (2,3), (2,4), (3,5), (4,5),(4,6), (4,5), (5,8), (6,7), (7,8)

(2) Cuthill McKee: Level sets: $\{1\}, \{2, 4\}, \{3, 5, 6, 7\}, \{8\}$ leads to the sequence 1, 2, 4, 3, 6, 5, 7, 8 and to new numbering

```
1                   5
| \               / |
|   \           /   |
|     \       /     |
|       \   /       |
2---------3---------6
|         |         |
|         |         |
|         |         |
|         |         |
4---------7---------8
```

(3) We need three colors because

```
    red                 r
    | \               / |
    |   \           /   |
    |     \       /     |
    |       \   /       |
black-------green-------b
    |         |         |
    |         |         |
    |         |         |
    |         |         |
    g---------r---------g
```

and leads to numbering (first red, then black, then green):

```
    1                   3
    | \               / |
    |   \           /   |
    |     \       /     |
    |       \   /       |
    4---------6---------5
    |         |         |
    |         |         |
    |         |         |
    |         |         |
    7---------2---------8
```

(4) Dissection: Shortest cut is edge (4,5) in original numbering. Leads to

```
    1                   4
    | \               / |
    |   \           /   |
    |     \       /     |
    |       \   /       |
    2---------7---------5
    |         |         |
    |         |         |
    |         |         |
    |         |         |
    3---------8---------6
```

and new pattern (1,2), (1,7), (2,3), (2,7), (3,8), (4,5), (4,7), (5,6), (5,7), (6,8), (7,8)