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Parallel Numerics

Exam

A handwritten sheet of paper (size A4, front and back page) may be used during the exam as mnemonic as well as the MPI operation reference distributed during the tutorials. No other material is allowed. The exam is to be solved within 90 minutes, and the answers are to be written in German or English. Please try to answer all parts of the questions precisely and briefly. For passing the exam you will need 17 out of 40 credits. The exam consists of 7 problems on 4 pages.

Problem 1: Miscellaneous (≈ 8 credits)

a) Describe the SPMD paradigm with a comparison to the SIMD and MIMD architecture classifications according to Michael J. Flynn. Refer also to synchronization of operations/instructions.

b) Describe briefly the difference between OpenMP and the Message Passing Interface (MPI)? Give 1 advantage and 1 disadvantage for each of them.

c) Given is the following C code fragment using MPI. Describe in detail wheather the code deadlocks or runs to completion.

```
...
int myrank;MPI_Request request;MPI_Status status;double a[8],b[8];
MPI_Init(&argc,&argv);
MPI_Comm_rank(MPI_COMM_WORLD,&myrank);
if( myrank == 0 ) {
    MPI_Irecv( b,8,MPI_DOUBLE,1,3,MPI_COMM_WORLD,&request );
    MPI_Send( a,8,MPI_DOUBLE,1,4,MPI_COMM_WORLD );
    MPI_Wait( &request,&status );
}
else if( myrank == 1 ) {
    MPI_Irecv( b,8,MPI_DOUBLE,0,4,MPI_COMM_WORLD,&request );
    MPI_Send( a,8,MPI_DOUBLE,0,3,MPI_COMM_WORLD );
    MPI_Wait( &request,&status );
}
MPI_Finalize();
...
```

d) The solution of a linear system $Ax = b$ with a matrix $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ can be obtained by using the SOR method. Here, the solution x is computed iteratively as a series $x^{(k)}$:

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) + (1-\omega) x_i^{(k)} \qquad (1)$$

for $i = 1, \ldots, n,\ k = 0, 1, \ldots$

For $\omega = 1$ SOR coincides with the Gauss-Seidel method. Change the given SOR formula (1) such that it describes the Jacobi method!

Problem 2: Parallel evaluation of arithmetic expressions **(≈ 5 credits)**
Consider the arithmetic expression

$$(b_1 \cdot b_2 \cdot b_3 + b_4 + b_5) \cdot b_6 + b_7 - b_8, \quad b_i \in \mathbb{R}. \tag{2}$$

a) How many time steps are necessary to evaluate (2) with its basic operations (addition, multiplication, subtraction, ...) on $p = 1$, $p = 2$, and $p = 3$ processors? Do **not!** modify the expression but use exactly the form as presented in (2). Show **and** explain the optimal evaluation scheme to proof your answer.

b) Is it possible to reduce the number of time steps by modifying the arithmetic expression given in (2) when using $p = 3$ processors? Proof your answer!

Problem 3: Storage costs for Matrix-Matrix multiplication **(≈ 4 credits)**
Consider the matrix-matrix product $C = A \cdot B$ where A, B, $C \in \mathbb{R}^{n \times n}$. To compute C, a 2D parallel algorithm can be used on a $\sqrt{p} \times \sqrt{p}$ processor grid, where p is the number of processors used. Each processor contains one block of each matrix. You may assume that p is a perfect square and that $\sqrt{p}$ divides n.
Example: for $p = 9$ the submatrices are distributed according to

$$\left(\begin{array}{c|c|c} A_{11}, B_{11}, C_{11} & A_{12}, B_{12}, C_{12} & A_{13}, B_{13}, C_{13} \\ \hline A_{21}, B_{21}, C_{21} & A_{22}, B_{22}, C_{22} & A_{23}, B_{23}, C_{23} \\ \hline A_{31}, B_{31}, C_{31} & A_{32}, B_{32}, C_{32} & A_{33}, B_{33}, C_{33} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} p_1 & p_2 & p_3 \\ \hline p_4 & p_5 & p_6 \\ \hline p_7 & p_8 & p_9 \end{array}\right).$$

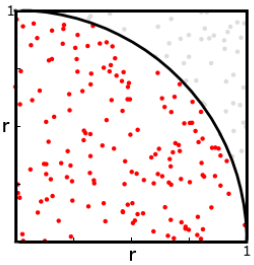
What is the total storage per processor required for each of the following algorithms in a) and b)? You should include storage for the matrices and the message buffers necessary for the communication. Give a short explanation and the formula dependent on p and n. *Hint:* for instace, each processor will require $\frac{n^2}{p}$ storage for its A block.

a) Straightforward algorithm using broadcasts along both rows and columns of processors (i.e., p_1 will broadcast its blocks to p_2, p_3, p_4, and p_7)?

b) Cannon's algorithm?

Problem 4: Calculation of π with Monte Carlo and MPI **(≈ 6 credits)**

Consider a quarter circle with a radius $r = 1$ which is within a square in the first quadrant, with corners at $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. See also Figure to the right. By choosing random points within the square (analogy of throwing darts 'randomly' at a dart board), we are able to calculate whether or not each point is within the quarter circle or not. The ratio of the area of the quarter circle to the area of the square is given by



$$\frac{\text{Area of quarter circle}}{\text{Area of square}} = \frac{\frac{1}{4} r^2 \pi}{r^2} \overset{r=1}{=} \frac{\pi}{4}. \tag{3}$$

(3) can be estimated by randomly choosing points (x, y) within the square and keeping track of the number of those points that fall within the quarter circle (those where $x^2 + y^2 <= 1$) versus the total number of points tried. Hence, π can be computed approximately by

$$\pi \approx \frac{4 \cdot \text{number of darts inside quarter circle}}{\text{total number of darts}} =: \frac{n_{\text{count}}}{n_{\text{darts}}} \quad \text{for} \quad n_{\text{darts}} \gg 1. \tag{4}$$

Given is the following C/MPI code fragment which stores the number of points which fall into the quarter circle in the variable `my_count`. According to the following questions, your code fragment should fit into the line 22.

```
7:  ...
8:  int my_id, nprocs, my_count = 0, count = 0, tag = 1;
9   int ndarts = 100000, master = 0;
10:  double x,y,z,pi;
11: MPI_Status status;
12: MPI_Init(&argc,&argv);
13: MPI_Comm_size(MPI_COMM_WORLD, &nprocs);
14: MPI_Comm_rank(MPI_COMM_WORLD, &my_id);
15:
16: for (int i = 0; i < ndarts; i++) {
17:     x = get_random_nmb();
18:     y = get_random_nmb();
19:     z = x*x + y*y;
20:     if (z <= 1) my_count++;
21: }
22:
23: // fill in your code here ...
24:
25: MPI_Finalize();
26: ...
```

a) The master process (rank 0) sums up the local `my_count`'s from all other processors (slaves) and computes π according to (4). Use **only!** the standard `MPI_Send()` and `MPI_Recv()` commands for the interprocess communication. Your code fragment should fit into line 22 of the above source code.

b) Now every processor computes its own approximation to π. The master process sums up all these local results and computes the average value of π. Use an appropriate MPI function which enables to 'collect' all the local results. Write your code fragment which fits in line 22 of the above source code.

Problem 5: Sparse Approximate Inverses **(≈ 5 credits)**

The SAI algorithm computes a sparse matrix $M \in \mathbb{R}^{n\times n}$ which approximates $A \in \mathbb{R}^{n\times n}$ inversely, i.e., $M \approx A^{-1}$. It uses the ansatz

$$\min_{M\in\mathcal{P}} \|AM - I\|_F^2, \tag{5}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and which requires an a priori chosen sparsity pattern $\mathcal{P}$ for M, i.e., one has to define the positions of the nonzeros in M a priori (by a given boolean matrix or sparsity set).

a) What is the inherent advantage of using the Frobeniusnorm approach according to (5)? Emphasize your answer with a formula corresponding to (5).

b) Compute the approximate inverse for the following system and preconditioner pattern:

$$\min_{M\in\mathcal{P}} \left\| \begin{pmatrix} 4 & 0 & -2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\|_F^2 .$$

Problem 6: LU decomposition **(≈ 6 credits)**

For matrices A we want to compute an LU decomposition on a parallel computer. The algorithm uses a block decomposition of the following form

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & * \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & * \\ 0 & U_{22} & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{6}$$

and is based on two subproblems. The computation is to be performed column-wise for L and U.

a) Give the formulae and a short description to compute the first columns of the LU decomposition in (6).

b) Based on results for the first columns in a), give the formulae and a short description for the computation of the second columns of the LU decomposition in (6).

c) Describe the approach in general how to compute the LU decomposition.

Problem 7: Graphs and colouring **(≈ 6 credits)**

Given is the symmetric update

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \end{pmatrix} = Ax = \frac{1}{5} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} x.$$

where

$$x^{(k+1)} = f(x^{(k)}) \tag{7}$$

reflects the dependency between the update $x^{(k+1)}$ and the old solution $x^{(k)}$ in an iterative (e.g., Gauss-Seidel) algorithm. $A \in \mathbb{R}^{6\times 6}$ and $x \in \mathbb{R}^6$. Consider to overwrite the vector $x^{(k)}$ immediately.

a) Give the corresponding graph of f which describes the data dependency in the update (7), i.e. give $G(f)$.

b) Give and explain a minimum colouring of $G(f)$ such that the update can be performed in parallel. How many steps have to be performed in parallel for the update?

c) Describe a renumbering of $G(f)$ according to the minimum coloring of $G(f)$ where the vertices of one color are numbered consecutively. Apply this permutation to rows and columns of A and give the resulting permuted matrix $\tilde{A}$.

d) What is the advantage of this reordering when using Gauss-Seidel? How many components x_j can be updated simultaneously?