Technische Universität München
Institut für Informatik
Dr. rer. nat. Tobias Neckel
Matous Sedlacek

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Parallel Numerics

Exercise 8: Stationary Methods

1) Stationary Methods Revisited

To solve the equation system $Ax = b$ stationary methods split up the matrix A into $A = M - N$:

$$\begin{aligned} Ax &= b \\ (M-N)x &= b \\ Mx &= Nx + b \\ Mx^{(n+1)} &= Nx^{(n)} + b \end{aligned}$$

i) Give the Richardson, Jacobi and Gauß–Seidel method using matrix notation.
Consider splitting $A = A - I + I$ with identity I:

$$\begin{aligned} Ax &= b \\ (A - I + I)x &= b \\ (A - I)x + x &= b \\ x &= b - (A - I)x \\ x &= (I - A)x + b \end{aligned}$$

With $M = I$ and $N = I - A$ the Richardson method is $x^{(n+1)} = (I - A)x^{(n)} + b$.
Consider splitting $A = D - L - U$ where D is diagonal part of A, L is strict lower and U is strict upper triangular part, respectively:

$$\begin{aligned} (D - L - U)x &= b \\ Dx - (L + U)x &= b \\ Dx &= b + (L + U)x \\ x &= D^{-1}(L + U)x + D^{-1}b \end{aligned}$$

With $M = D$ and $N = L + U$ the Jacobi method is $x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b$.
Consider splitting $A = D - L - U$:

$$(D - L - U)x = b$$
$$(D - L)x - Ux = b$$
$$x = (D - L)^{-1}Ux + (D - L)^{-1}b$$

With $M = D - L$ and $N = U$ the Gauß-Seidel method is $x^{(n+1)} = (D - L)^{-1}Ux^{(n)} + (D - L)^{-1}b$.

ii) Give the Richardson, Jacobi and Gauß–Seidel method in pseudo code and identify GAXPYs, SAXPYs, ...

Richardson:

while not converged

for $j = 1...n$

$$x_j^{(n+1)} = \underbrace{\sum_{i=1}^{n}(-a_{ji}x_i^{(n)})}_{scalar\ \ product} + x_j^{(n)} + b_j$$

Pseudocode for Richardson with SAXPY for inner for-loop and GAXPY for outer for-loop:

$x^{(n+1)} = 0$

for $j = 1, ..., n$

for $i = 1, ..., n$

$x_i^{(n+1)} = x_i^{(n+1)} - x_j^{(n)}a_{ij}$

end

end

$x^{(n+1)} = x^{(n+1)} + x^{(n)} + b$

Jacobi:

while not converged

for $j = 1, ..., n$

$$x_j^{(n+1)} = \tfrac{1}{a_{jj}}\left(\sum_{i=1,\ \ i\neq j}^{n}(-a_{ji}x_i^{(n)}) + b_j\right)$$

Pseudocode for Jacobi with SAXPY for inner for-loop and GAXPY for outer for-loop:

$x^{(n+1)} = 0$

for $j = 1, ..., n$

for $i = 1, ..., n$

if $i \neq j$

$x_i^{(n+1)} = x_i^{(n+1)} - x_j^{(n)}a_{ij}$

end

end

end

for $i = 1, ..., n$

$x_i^{(n+1)} = \frac{x_i^{(n+1)} + b_i}{a_{ii}}$

end

Gauß-Seidel:
while not converged
for $j = 1, ..., n$

$$x_j^{(n+1)} = \frac{1}{a_{jj}} \left(\sum_{i=j+1}^{n} (-a_{ji} x_i^{(n)}) + b_j + \sum_{i=1}^{j-1} (-a_{ji} x_i^{(n+1)}) \right)$$

In matrix notation the solution $x^{(n+1)}$ *is obtained by solving*

$$(D - L)x^{(n+1)} = \underbrace{Ux^{(n)}}_{GAXPY} + b.$$

Pseudocode for Gauß-Seidel. No SAXPYs in this formulation:

$x^{(n+1)} = 0$
for $j = 1, ..., n$
$c = b_j$
for $i = j + 1, ..., n$
$c = c - x_i^{(n)} a_{ji}$
end
for $i = 1, ..., j - 1$
$c = c - x_i^{(n+1)} a_{ji}$
end
$x_j^{(n+1)} = \frac{c}{a_{jj}}$
end

GAXPY available if $Ux^{(n)}$ *is computed in a first step completely on its own. Afterwards the triangular solve is performed.*

iii) For the weighted relaxation schemes, one scales the iteration rule above with a factor ω and adds the trivial iteration $x^{(n+1)} = x^{(n)}$. Derive the $\omega - JAC$ and SOR (Successive-Over-Relaxation) scheme in matrix notation.

$\omega - JAC$ *relaxation scheme by using relaxation parameter* ω *in convex combination:*

$$\begin{aligned}
x^{(n+1)} &= (1 - \omega)x^{(n)} + \omega x^{(n+1)} \\
x^{(n+1)} &= (1 - \omega)x^{(n)} + \omega D^{-1}(L + U)x^{(n)} + \omega D^{-1}b \\
x^{(n+1)} &= \underbrace{[(1 - \omega)I + \omega D^{-1}(L + U)]}_{=:R_\omega} x^{(n)} + \omega D^{-1}b \\
x^{(n+1)} &= R_\omega x^{(n)} + \omega D^{-1}b
\end{aligned}$$

SOR scheme as a variant of relaxed Gauß-Seidel:

$$
\begin{aligned}
\omega(D-L-U)x &= \omega b \\
\omega(D-L-U)x + Dx &= \omega b + Dx \\
\omega(D-U)x - \omega Lx + Dx &= \omega b + Dx \\
Dx - \omega Lx &= \omega b + Dx - \omega(D-U)x \\
(D-\omega L)x &= Dx - \omega(D-U)x + \omega b \\
(D-\omega L)x &= [(1-\omega)D + \omega U]x + \omega b \quad \textit{and thus the iteration} \\
x^{(n+1)} &= \underbrace{(D-\omega L)^{-1}[(1-\omega)D + \omega U]}_{=:S_\omega} x^{(n)} + (D-\omega L)^{-1}\omega b \\
x^{(n+1)} &= S_\omega x^{(n)} + (D-\omega L)^{-1}\omega b. \qquad (1)
\end{aligned}
$$

2) Residual–based Notation

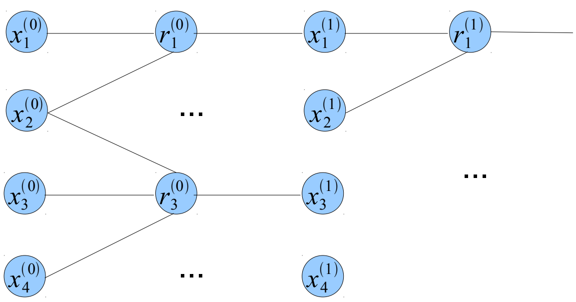
The residual is defined as

$$r = b - Ax$$

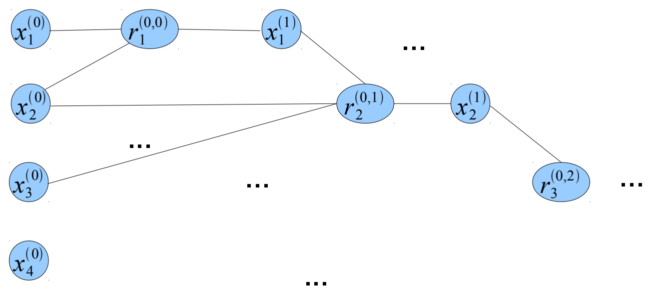
i) Give the Richardson, Jacobi and Gauß–Seidel method using the residual.
In general: $x^{(n+1)} = x^{(n)} + M^{-1}r^{(n)}$.
Richardson with residual: $x^{(n+1)} = x^{(n)} + r^{(n)}$
Jacobi with residual: $x^{(n+1)} = x^{(n)} + D^{-1}r^{(n)}$
Gauß-Seidel with residual: $x^{(n+1)} = x^{(n)} + (D-L)^{-1}r^{(n)}$

ii) Give the $\omega - JAC$ and SOR scheme using the residual.
In general: $x^{(n+1)} = x^{(n)} + \omega M^{-1}r^{(n)}$.
$\omega - JAC$ *with residual:* $x^{(n+1)} = x^{(n)} + \omega D^{-1}r^{(n)}$
SOR scheme with residual: $x^{(n+1)} = x^{(n)} + \omega(D-\omega L)^{-1}r^{(n)}$

iii) Give a scetch of the data dependendy graph for both computing the residual and updating the solution according to the Jacobi and the GS scheme. (To simplify matters: Assume that A is tridiagonal)
Scetch of data dependency between x and r for Jacobi:



Scetch of data dependency between x and r for Gauß-Seidel with residual formulation $x_j^{(n+1)} = x_j^{(n)} + (D-L)^{-1}r_j^{(n,j-1)}$ *where $j-1$ updates are considered in r_j:*



iv) Which parallel algorithms for matrix vector products do you know (already)?
In the residual notation all solvers are reduced to matrix-vector-products. Thus, one can use e.g. Cannon's algorithm, cyclic assignement, etc..

3) SOR Implementation

In this section, we want to implement the SOR. As simplification for the following algorithms we assume that the iteration index k is always running from 0 to a maximum number k_{stop}. With the definitions

$$\alpha_i := \frac{\omega}{a_{ii}} \quad \text{for} \quad i = 1, \ldots, n \qquad \text{and} \qquad b_{ij} := \begin{cases} -a_{ij} & \text{for} \quad i \neq j \\ \frac{1-\omega}{\omega} a_{ii} & \text{for} \quad i = j \end{cases}$$

a serial algorithm for the SOR method can be given as follows:

for $k = 0$ to k_{stop}
  for $i = 1$ to n
    $s := d_i$
    for $j = 1$ to n
      $s := s + b_{ij} x_j$ $(*)$
    $x_i := \alpha_i s$

A parallel algorithm can be implemented in a similar way: The b_{ij} are distributed columnwise on p processors in a cyclic way (cp. the Parallel Gauss Elimination, Exercise 6). Every processor calculates only a part of the sum in $(*)$. Following, the x_i are calculated successively on different processors after receiving the parts of the sum form the other processors. Suitable communication is necessary. The parallel algorithm has the following shape:

for $k = 0$ to k_{stop}
  for $i = 1$ to n
    $a := \sum_{j \in mycolumns} b_{ij} x_j$
    if $i \in mycolumns$
      Get a from all other processors and calculate $s := \sum_p a$
      $x_i := \alpha_i (s + d_i)$

Implement the serial and parallel algorithm! On which processor do you find the solution x after running the parallel algorithm? Can you observe a speedup?

See sourcecode to corresponding tutorial on webpage.