1 Further notes on Hockney/Golub

- (Parallelizable) approach to solve tridiagonal linear systems of equations.
- Entries (sub- and superdiagonal) are moving away from diagonal.
- Can be combined with cyclic reduction to obtain a divide and conquer principle: The original system breaks in to 2 independant tridiagonal systems, that can be solved independently.
- Cyclic Reduction (CR):
  - Requires $log(n)$ steps, each $\mathcal{O}(n)$ operations. Total work of $\mathcal{O}(nlog(n))$.
  - Sequentially, CR inferior to LU or Cholesky decompostion, which require only $\mathcal{O}(n)$ work for tridiagonal system.
- Problems for H/G: $b_i^{(k)}$ must not be zero! (Division by zero). This condition is not satisfied automatically for arbitrary nonsingular (invertible) tridiagonal matrices. Even when Gaussian elimination without pivoting is possible, H/G is not necessarily applicable. E.g., for the following exemplary matrix:

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 1 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 1 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

- But: If A is H-matrix then H/G is possible for every right-hand side d.
- H-matrix: $A \in \mathbb{C}^{n\times n}$ is H-matrix if comparison matrix $M(A) = \alpha_{ij} \in \mathbb{R}^{n\times n}$ is M-matrix, where entries of $M(A)$ are $\alpha_{ij} := |a_{ii}|$ for $i = j$ and $\alpha_{ij} := -|a_{ij}|$ otherwise.
- M-matrix: $A \in \mathbb{R}^{n\times n}$ is M-matrix if
  - A is nonsingular
  - if non-positive values are exlusively off the main diagonal, i.e., $a_{ij} \leq 0$ for $i \neq j$
  - and if the inverse A^{-1} has only non-negative entries.

  Note: Furthermore, inverse monotony holds for M-matrices: A is M-matrix and $x, y \in \mathbb{R}^n$ then $Ax \leq Ay \Rightarrow x \leq y$ where the relation $\leq$ is taken componentwise. See Example M-Matrix.

$$\begin{pmatrix} 10 & -1 & -4 & & \\ -1 & 10 & -1 & -4 & \\ -4 & -1 & 10 & -1 & -4 \\ & -4 & -1 & 10 & -1 \\ & & -4 & -1 & 10 \end{pmatrix}$$

- Parallelization of H/G. Consider case $n = p$ (number of pes equals number or rows):

  - p_i computes $a_i^{(k)}$, $b_i^{(k)}$, $c_i^{(k)}$, $d_i^{(k)}$ $\rightarrow$ communication necessary for $a_j^{(k-1)}$, $b_j^{(k-1)}$, $c_j^{(k-1)}$, $d_j^{(k-1)}$, $j \in \{i-2^{k-1}, i, i+2^{k-1}\}$. For $j = i - 2^{k-1}$ communication with $p_{i-2^{k-1}}$ necessary, for $j = i + 2^{k-1}$ communication with $p_{i+2^{k-1}}$ necessary.
  - per iteration every p_i is working / active
  - After computation of step $k = N$, p_i will compute $x_i = \frac{d_i^{(N)}}{b_i^{(N)}}$
- Parallelization of H/G for case $n = pq$ (processor has block of rows)? Think about it!